The Riemann Hypothesis

Frank Vega

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Abstract. Let's define $\delta(x) = (\sum_{q \leq x} \frac{1}{q} - \log \log x - B)$, where $B \approx 0.2614972128$ is the Meissel-Mertens constant. The Robin theorem states that $\delta(x)$ changes sign infinitely often. Let's also define $S(x) = \theta(x) - x$, where $\theta(x)$ is the Chebyshev function. A theorem due to Erhard Schmidt implies that $S(x)$ changes sign infinitely often. Using the Nicolas theorem, we prove that when the inequalities $\delta(x) \leq 0$ and $S(x) \geq 0$ are satisfied for some $x \geq 127$, then the Riemann Hypothesis should be false. However, the Mertens second theorem states that $\lim_{x \to \infty} \delta(x) = 0$. Moreover, a result from the Grönwall paper could be restated as $\lim_{x \to \infty} S(x) = 0$. In this way, this work could mean a new step forward in the direction for finally solving the Riemann Hypothesis.

1 Introduction

In mathematics, the Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$ [1]. Let $N_n = 2 \times 3 \times 5 \times 7 \times 11 \times \cdots \times p_n$ denotes a primorial number of order n such that p_n is the n^{th} prime number. Say $\mathsf{Nicolas}(p_n)$ holds provided

$$\prod_{q \mid N_n} \frac{q}{q-1} > e^{\gamma} \times \log \log N_n.$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, log is the natural logarithm, and $q \mid N_n$ means the prime number q divides to N_n. The importance of this property is:

Theorem 1.1 *[6], [7].* $\mathsf{Nicolas}(p_n)$ *holds for all prime number $p_n > 2$ if and only if the Riemann Hypothesis is true.*

In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

where $p \leq x$ means all the prime numbers p that are less than or equal to x. We use the following property of the Chebyshev function:

Theorem 1.2 *[9]. For $x > 1$:*

$$\theta(x) = (1 + \varepsilon(x)) \times x$$

where $\varepsilon(x) < \frac{1}{2 \times \log x}$.

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Let's define $S(x) = \theta(x) - x$. In the Grönwall paper appears this:

Theorem 1.3 *[3].*

$$\lim_{x\to\infty} \frac{\theta(x)}{x} = 1,$$

and that could be restated as:

$$\lim_{x\to\infty} S(x) = 0.$$

Nicolas also proves that

Theorem 1.4 *[7]. For $x \geq 121$:*

$$\log\log\theta(x) \geq \log\log x + \frac{S(x)}{x \times \log x} - \frac{S(x)^2}{x^2 \times \log x}.$$

From the paper of Schmidt, then we can deduce that:

Theorem 1.5 *[10]. $S(x)$ changes sign infinitely often.*

The famous Mertens paper provides the statement:

Theorem 1.6 *[5].*

$$\log\left(\prod_{q\leq x} \frac{q}{q-1}\right) = \sum_{q\leq x} \frac{1}{q} + \gamma - B - \frac{1}{2} \times \sum_{q>x} \frac{1}{q^2} - \frac{1}{3} \times \sum_{q>x} \frac{1}{q^3} - \cdots$$

where $B \approx 0.2614972128$ is the Meissel-Mertens constant.

Let's define:

$$\delta(x) = \left(\sum_{q\leq x} \frac{1}{q} - \log\log x - B\right),$$

Robin theorem states the following result:

Theorem 1.7 *[8]. $\delta(x)$ changes sign infinitely often.*

In addition, the Mertens second theorem states that:

Theorem 1.8 *[5].*

$$\lim_{x\to\infty} \delta(x) = 0.$$

Putting all together yields the proof that when the inequalities $\delta(x) \leq 0$ and $S(x) \geq 0$ are satisfied for some $x \geq 127$, then the Riemann Hypothesis should be false.

2 Central Lemma

Lemma 2.1 *For $x \geq 127$:*

$$\frac{S(x)}{x} < 1.$$

Proof By the theorem 1.2, $\forall x \geq 127$:

$$\begin{aligned}\frac{S(x)}{x} &= \frac{\theta(x) - x}{x} \\ &= \frac{(1 + \varepsilon(x)) \times x - x}{x} \\ &= \frac{x \times ((1 + \varepsilon(x)) - 1)}{x} \\ &= (1 + \varepsilon(x) - 1) \\ &= \varepsilon(x) \\ &< \frac{1}{2 \times \log x} \\ &< 1.\end{aligned}$$

■

3 Main Theorem

Theorem 3.1 *If the inequalities $\delta(x) \leq 0$ and $S(x) \geq 0$ are satisfied for some $x \geq 127$, then the Riemann Hypothesis should be false.*

Proof For some $x \geq 127$, suppose that simultaneously $\mathsf{Nicolas}(p)$ holds and the inequalities $\delta(x) \leq 0$ and $S(x) \geq 0$ are satisfied, where p is the greatest prime number such that $p \leq x$. If $\mathsf{Nicolas}(p)$ holds, then

$$\prod_{q \leq x} \frac{q}{q-1} > e^{\gamma} \times \log \theta(x).$$

We apply the logarithm to the both sides of the inequality:

$$\log \left(\prod_{q \leq x} \frac{q}{q-1} \right) > \gamma + \log \log \theta(x).$$

We use that theorem 1.6:

$$\log \left(\prod_{q \leq x} \frac{q}{q-1} \right) = \sum_{q \leq x} \frac{1}{q} + \gamma - B - \frac{1}{2} \times \sum_{q > x} \frac{1}{q^2} - \frac{1}{3} \times \sum_{q > x} \frac{1}{q^3} - \cdots .$$

Besides, we use that theorem 1.4:

$$\log \log \theta(x) \geq \log \log x + \frac{S(x)}{x \times \log x} - \frac{S(x)^2}{x^2 \times \log x}.$$

Putting all together yields the result:

$$\sum_{q\leq x}\frac{1}{q}+\gamma-B-\frac{1}{2}\times\sum_{q>x}\frac{1}{q^2}-\frac{1}{3}\times\sum_{q>x}\frac{1}{q^3}-\cdots$$
$$>\gamma+\log\log\theta(x)$$
$$\geq\gamma+\log\log x+\frac{S(x)}{x\times\log x}-\frac{S(x)^2}{x^2\times\log x}.$$

Let distribute it and remove γ from the both sides:

$$\sum_{q\leq x}\frac{1}{q}-\log\log x-B-\frac{1}{2}\times\sum_{q>x}\frac{1}{q^2}-\frac{1}{3}\times\sum_{q>x}\frac{1}{q^3}-\cdots>$$
$$\frac{1}{\log x}\times\left(\frac{S(x)}{x}-\frac{S(x)^2}{x^2}\right).$$

We know that $\delta(x)=\sum_{q\leq x}\frac{1}{q}-\log\log x-B$. Moreover, we know that

$$\left(\frac{S(x)}{x}-\frac{S(x)^2}{x^2}\right)\geq 0.$$

Certainly, according to the lemma 2.1, we have that $\frac{S(x)}{x}<1$. Consequently, we obtain that $\frac{S(x)}{x}\geq\frac{S(x)^2}{x^2}$ under the assumption that $S(x)\geq 0$, since for every real number $0\leq x<1$, the inequality $x\geq x^2$ is always satisfied. To sum up, we would have that

$$\delta(x)-\frac{1}{2}\times\sum_{q>x}\frac{1}{q^2}-\frac{1}{3}\times\sum_{q>x}\frac{1}{q^3}-\cdots>0$$

because of

$$\frac{1}{\log x}\times\left(\frac{S(x)}{x}-\frac{S(x)^2}{x^2}\right)\geq 0.$$

However, the inequality

$$\delta(x)-\frac{1}{2}\times\sum_{q>x}\frac{1}{q^2}-\frac{1}{3}\times\sum_{q>x}\frac{1}{q^3}-\cdots>0$$

is never satisfied when $\delta(x)\leq 0$. By contraposition, $\mathsf{Nicolas}(p)$ does not hold when $\delta(x)\leq 0$ and $S(x)\geq 0$ are satisfied for some $x\geq 127$, where p is the greatest prime number such that $p\leq x$. In conclusion, if $\mathsf{Nicolas}(p)$ does not hold for some prime number $p\geq 127$, then the Riemann Hypothesis should be false due to the theorem 1.1. ■

4 Discussion

The Riemann Hypothesis has been qualified as the Holy Grail of Mathematics [4]. It is one of the seven Millennium Prize Problems selected by the Clay Mathematics Institute to carry a US 1,000,000 prize for the first correct solution [2]. In the theorem 3.1, we show that if the inequalities $\delta(x)\leq 0$ and

$S(x) \geq 0$ are satisfied for some $x \geq 127$, then the Riemann Hypothesis should be false. Nevertheless, the well-known theorem 1.8 states that

$$\lim_{x\to\infty} \delta(x) = 0.$$

In addition, the theorem 1.3 states that

$$\lim_{x\to\infty} S(x) = 0.$$

Indeed, we think this work could help to the scientific community in the global efforts for trying to solve this outstanding and difficult problem.

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CopSonic, 1471 Route de Saint-Nauphary 82000 Montauban, France
Email: vega.frank@gmail.com