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Abstract. In time series analysis, comparing spectral densities of several processes with almost periodic spectra is an interested problem. The aim of this paper is to give an approach to test the equality among spectral densities of several independent almost periodically correlated (cyclostationary) processes. This approach is based on the limiting distribution for the periodogram and the discrete Fourier transform. The simulation results indicate that the approach well acts.

Key words: Almost periodically correlated processes, Almost cyclostationary processes, Spectral density.

1. Introduction

Comparing spectral densities of several processes is an important topic that has many applications in economics, finance, physics, signal processing, and many others. The researchers like to explore if several time series have the same stochastic mechanism.

Many references have considered the comparison, classification and clustering of two or several processes. For example, De Souza and Thomson (1982), Coates and Diggle (1986), Potscher and Reschenhofer (1988), Diggle and Fisher (1991), Dargahi- Noubary (1992), Diggle and al Wasel (1997), Kakizawa et al. (1998), Timmer et al. (1999), Maharaj [(1999); (2000); (2002); (2005)], Caiado et al. (2006), Eichler (2008), Fokianos and Savvides (2008), Caiado et al. (2009), Dette and Paparoditis (2009), Dette et al. (2010), Dette and Hildebrandt (2011), Jentsch (2012), Jentsch and Pauly (2012), Salcedo et al. (2012), Jentsch and Pauly (2014), Triacca (2016), Mahmoudi et al. (2017) studied these subjects for stationary time series.

But the stationarity assumption is not satisfied in many situations, specially when the processes have a periodic rhythm. In these cases, cyclostationary (CS) and almost cyclostationary (ACS) processes are naturally applied to model the rhythmic component. Mahmoudi et al. (2018b) and Mahmoudi et al. (2018c) considered the comparison, classification and clustering of two CS time series, respectively.

The ACS is a large non-stationary time series class that contained stationary and CS processes. The mean and auto-covariance functions of ACS are almost periodic. The spectra of these processes are supported on lines that are parallel to the main diagonal, $T_j(x) = x \pm \alpha_j, j = 1, 2, \ldots$, in spectral square $[0,2\pi) \times [0,2\pi)$. The theories and applications of ACS time series were considered in many researches such as Gladyshev [(1961); (1963)], Gardner (1991), Hurd (1991), Hurd and Leskow (1992), Leskow and Weron (1992), Gardner (1994), Leskow (1994), Lii and Rosenblatt [(2002); (2006)], Gardner et al. (2006), Hurd and Miamee (2007), Lenart [(2008); (2011)], Napolitano (2012), Lenart (2013), Lenart and Pipien [(2013a), (2013b)], Mahmoudi et al. (2015), Mahmoudi and Maleki (2017), Nematollahi et al. (2017), Lenart and Pipien (2017), and Mahmoudi et al. (2018a).

In this work, the asymptotic distribution for the periodogaram and discrete Fourier transform of ACS processes will be applied to construct an approach to compare and classify several ACS processes. Section 2 is devoted to notations and preliminaries. The technique to compare and classify the ACS processes is presented in Section 3. The ability of the introduced approach is studied by means of extensive Monte Carlo simulations, and real world problem, in Sections 4 and 5, respectively.

2. Notations and Preliminaries

Definition 1: Almost Periodic Function [Corduneanu (1989)]

A function $f(t): Z \to R$ is almost periodic if for any $\varepsilon > 0$, there exists a positive integer L_ε such that among any $L_\varepsilon > 0$ consecutive integers there is a positive integer p_ε such that

$$sup_{t \in \mathrm{Z}} |f(t + p_\varepsilon) - f(t)| < \varepsilon.$$

Definition 2: ACS Process [Mahmoudi et al. (2018a)]

A second order process $\{X_t: t \in Z\}$ is called ACS if the process has almost periodic mean, $\mu(t) = E(X_t)$, and autocovariance, $B(t, \tau) = cov(X_t, X_{t+\tau})$, at t, for every $\tau \in$ Z.

As Mahmoudi et al. (2018a), the following assumptions have been considered in this work:

(A1) $\{X_t: t \in Z\}$ *is a zero-mean and real-valued time series.*

(A2) X_t *is an ACS process.*

By these assumptions, the autocovariance function $B(t, \tau)$ can be represented by

$$B(t,\tau) \sim \sum_{\omega \in W_t} a(\omega,\tau) e^{i\omega t},$$

where

$$a(\omega,\tau) = \lim_{n\to\infty}\left(\frac{1}{n}\sum_{j=1}^{n} B(j,\tau)\, e^{-i\omega t}\right),$$

and for fixed τ. Also as Corduneanu (1989) and Hurd (1991) indicated the set $W_\tau = \{\omega \in [0,2\pi): a(\omega,\tau) \neq 0\}$ is a countable set of frequencies.

(A3) $W = \bigcup_{\tau\in Z} W_\tau$, is a finite set and the spectra of X_t is supported on lines that are parallel to the main diagonal, $T_j(x) = x \pm \alpha_j, j = 1, 2, \ldots,$ in spectral square $[0,2\pi) \times [0,2\pi)$. Thus we have

$$B(t,\tau) = \sum_{\omega\in W} a(\omega,\tau) e^{i\omega t},$$

and the spectral measure of X_t, will be supported on the set

$$S = \bigcup_{\omega\in W} \{(\nu,\gamma) \in [0,2\pi)^2 : \gamma = \nu - \omega\}.$$

Moreover, the coefficients

$$a(\omega,\tau) = \int_0^{2\pi} e^{i\xi\tau} r_\omega(d\xi),$$

are the Fourier transforms of the measures $r_\omega(\cdot)$.

We note that the $r_\omega$will be identified if the spectral measure of X_t be restricted on the line $\gamma = \nu - \omega$, modulo 2π, where $\omega \in W$.
Remark: In the rest of paper, all equalities of frequencies are modulo 2π.

(A4) r_0 is an absolute continuous measure with respect to the Lebesgue measure.

Dehay and Hurd (1994) shown by considering this assumption and $\sum_{\tau=-\infty}^{\infty} |a(\omega,\tau)| < \infty$, for any $\omega \in W$, result in a spectral density function $f_\omega(\cdot)$ exists such that

$$f_\omega(\nu) = \frac{1}{2\pi}\sum_{\tau=-\infty}^{\infty} a(\omega,\tau) e^{-i\nu\tau}.$$

Consequently, an ACS process with support on a finite number of cyclic frequencies is represented by

$$X_t = \int_0^{2\pi} e^{-itx} \zeta(dx), \quad t \in \mathbb{Z},$$

where ζ is a random spectral measure on $[0, 2\pi)$ such that

$$E\left(\zeta(d\theta)\overline{\zeta(d\theta')}\right) = 0, (\theta, \theta') \notin S.$$

As Mahmoudi et al. (2018) indicated, the spectral distribution and density matrices of ζ, are defined by

$$\boldsymbol{F}(d\lambda) = \left[F_{k,j}(d\lambda)\right]_{j,k=1,\dots,m},$$

and

$$\boldsymbol{f}(\lambda) = \frac{d\boldsymbol{F}}{d\lambda} = \left[f_{k,j}(\lambda)\right]_{j,k=1,\dots,m},$$

respectively, where

$$F_{k,j}(d\lambda) = E\left(\zeta\,(d\lambda + \alpha_k)\overline{\zeta\,(d\lambda + \alpha_j)}\right), k, j = 1, \dots, m,$$

anf $f_{k,j}$ is spectral density correspond to $F_{k,j}$.

Definition 3: Discrete Fourier Transform (DFT)

Assume a sample $X_0, \dots, X_{N-1}$, from ACS process $\{X_t: t \in Z\}$. The DFT of the finite sequence $X_0, \dots, X_{N-1}$, is defined by

$$d_X(\lambda) = N^{-1/2}\sum_{t=0}^{N-1} X_t e^{-it\lambda}\ , \lambda \in [0,2\pi).$$

Definition 4: Periodogram

Assume that we have a sample $X_0, \dots, X_{N-1}$, from ACS process $\{X_t: t \in Z\}$. The DFT of the finite sequence $X_0, \dots, X_{N-1}$, is defined by

$$I_X(\lambda) = |d_X(\lambda)|^2\ , \lambda \in [0,2\pi).$$

The distribution of DFT and periodogram of ACS processes are widely studied by Lenart (2013), Lenart and Pipien [(2013a); (2017)] and Mahmoudi et al. (2018a).

3. Methodology

Suppose $\left\{X_t^{(1)}, t = 1, \dots, n_1\right\}$, $\left\{X_t^{(2)}, t = 1, \dots, n_2\right\}$, ..., $\left\{X_t^{(l)}, t = 1, \dots, n_l\right\}$, are *l* independent ACS processes with *m* spectral cycles.

Commonly, the researchers want to test the null hypothesis $H_0: X_t^{(1)} \equiv X_t^{(2)} \equiv \cdots \equiv X_t^{(l)}$, that is equivalent to $H_0: \boldsymbol{f_1} = \boldsymbol{f_2} = \cdots = \boldsymbol{f_l}$, such that $\boldsymbol{f_1}, \dots,$ and $\boldsymbol{f_l}$ are the spectral density matrices respectively corresponding to $X_t^{(1)}$, …, and $X_t^{(l)}$. If the null hypothesis H_0 is not accepted then it can be concluded that at least two time series of the l time series have different rhythms, and if H_0 is accepted consequently the stochastic behaviours of all processes are similar.

Mahmoudi et al. (2018a) introduced the periodogram for ACS time series as

$$\boldsymbol{I}_X^m(\lambda) = \boldsymbol{d}_X^m(\lambda)\boldsymbol{d}_X^{m*}(\lambda),$$

such that

$$\boldsymbol{d}_X^m(\lambda) = \left(d_X\left(T_1(\lambda)\right), d_X\left(T_2(\lambda)\right) \dots, d_X\left(T_m(\lambda)\right)\right)', \lambda \in [0, 2\pi),$$

where $\boldsymbol{d}_X^{m*}(\lambda)$ is the complex conjugate transpose of $\boldsymbol{d}_X^m(\lambda)$.

Lemma 3.1: Let $\{X_t,\ t \in \mathbb{Z}\}$ is an ACS time series with corresponding spectral density $\boldsymbol{f}(\lambda)$, $\lambda \in [0,2\pi)$. Assume $\lambda_1 < \cdots < \lambda_J$ are frequencies in $0,2\pi)$. Then

(i) $\boldsymbol{f}(\lambda)$ can be asymptotically estimated by $\hat{\boldsymbol{f}}(\lambda) := \frac{\boldsymbol{I}_X^m(\lambda)}{2\pi}$.

(ii) $\boldsymbol{d}_X^m(\lambda_j), j = 1, \dots, J$, have the asymptotic and independent m-variate complex normal distributions, $N_m^c\left(0, 2\pi\boldsymbol{f}(\lambda_j)\right)$.

(iii) $\boldsymbol{I}_X^m(\lambda_j), j = 1, \dots, J$, have the asymptotic and independent complex Wishart distributions, $W_m^c\left(\mathbf{1},\ 2\pi\boldsymbol{f}(\lambda_j)\right)$.

Proof: Mahmoudi et al. (2018a). ■

Let $Y_j^{(k)} = Re\left(\boldsymbol{d}_{X^{(k)}}^m(\lambda_j)\right), j = 1, \dots, J, k = 1,2, \dots, l,$ and $Z_j^{(k)} = Im\left(\boldsymbol{d}_{X^{(k)}}^m(\lambda_j)\right), j = 1, \dots, J, k = 1,2, \dots, l,$ such that $\boldsymbol{d}_{X^{(k)}}^m(\lambda_j)$, is the $\boldsymbol{d}_X^T(\lambda_j)$ corresponding to k^{th} time series. As a result of Lemma 3.1, it can be concluded that for $k = 1,2, \dots, l$, the asymptotic distribution of $W_j^{(k)} = \left(Y_j^{(k)}, Z_j^{(k)}\right)'$ is $N_{2m}\left(0, \boldsymbol{\Sigma}_j^{(k)}\right)$, such that $\boldsymbol{\Sigma}_j^{(k)} = \begin{bmatrix} \boldsymbol{V}_{Y_jY_j}^{(k)} & \boldsymbol{V}_{Y_jZ_j}^{(k)} \\ \boldsymbol{V}_{Z_jY_j}^{(k)} & \boldsymbol{V}_{Z_jZ_j}^{(k)} \end{bmatrix}$, $\boldsymbol{V}_{AB} = COV(A, B)$. Consequently, the asymptotic distribution of $U^{(k)} = \sum_{j=1}^{J} W_j^{(k)}$ is $N_{2m}\left(0, \boldsymbol{\Sigma}^{(k)}\right)$, such that $\boldsymbol{\Sigma}^{(k)} = \boldsymbol{\Sigma}_1^{(k)} + \cdots + \boldsymbol{\Sigma}_J^{(k)}$.

3.1. Test of Hypothesis

As previous discussion, usually, the researchers want to test the null hypothesis H_0: $\boldsymbol{f_1} = \boldsymbol{f_2} = \cdots = \boldsymbol{f_l}$, that is equivalent to H_0: $\boldsymbol{\Sigma}^{(1)} = \boldsymbol{\Sigma}^{(2)} = \cdots = \boldsymbol{\Sigma}^{(l)}$. In general, the Box's M test, Box [(1949); (1950)], is applied to compare the covariance matrices of m independent populations. As Rencher and Christensen (2012), firs we compute the modified likelihood-ratio statistic T_0 given by

$$T_0 = (N-m)Ln\left|\boldsymbol{S}_{pooled}\right| - \sum_{k=1}^{m}\left((N_k-1)Ln|\boldsymbol{S}_k|\right),$$

where N_k and $\boldsymbol{S}_k$ are sample size and sample covariance matrix of k^{th} sample, respectively, $N = N_1 + N_2 + \cdots + N_m$, and

$$\boldsymbol{S}_{pooled} = \sum_{k=1}^{m}(N_k-1)\,\boldsymbol{S}_k/(N-m).$$

Then the accurate χ^2 and F approximations (Rencher and Christensen (2012)) are computed by

Box's χ^2 approximation is given by

$$\chi_0^2 = (1-c_1)T_0,$$

and

$$F_0 = \begin{cases} b_1 T_0 & if\ c_2 > c_1^2 \\ \dfrac{-a_2 b_2 T_0}{a_1(1-b_2T_0)} & otherwise \end{cases}$$

which have approximately χ^2 distribution with $(m-1)m(2m+1)$ degrees of freedom and

F distribution with a_1 and a_2 degrees of freedom, respectively, such that

$$c_1 = \frac{8m^2+6m-1}{6(2m+1)(m-1)}\left(\sum_{k=1}^{m}(N_k-1)^{-1} - (N-m)^{-1}\right),$$

$$c_2 = \frac{(2m-1)(m+1)}{3(m-1)}\left(\sum_{k=1}^{m}(N_k-1)^{-2} - (N-m)^{-2}\right),$$

$$a_1 = (m-1)m(2m+1),\ \ a_2 = \frac{a_1+2}{|c_2-c_1^2|}\ , b_1 = \frac{1-c_1-\dfrac{a_1}{a_2}}{a_1}\ ,$$

and

$$b_2 = \frac{1-c_1+\dfrac{2}{a_2}}{a_2}.$$

Therefore the critical regions for a test of size α for the χ^2 and F approximations are

$$\chi_0^2 > \chi_\alpha^2(a_1),$$

and

$$F_0 > F_\alpha(a_1, a_2).$$

4. Simulation Study

To analyze the accuracy of proposed method, we generated

$$(n_1, n_2, n_3) = \{(100, 50,75), (150, 75,100), (200,150, 100), (500, 250,300)\}$$

observations from the ACS processes

$$X_t^{(i)} = (1 + \cos(\omega_i t))Y_t^{(i)}, \omega \in (0, \infty), \quad i = 1,2,3,$$

where

$$Y_t^{(i)} = Z_t^{(i)} + 0.5Z_{t-1}^{(i)},$$

and $Z_t^{(i)}, i = 1,2,3,$ are independent sequences of IIDN(0,1).

The spectral mass of $X_t^{(i)}$ is supported on the lines given by

$$T_1(x) = x, T_2(x) = x + \omega_i, T_3(x) = x - \omega_i, T_4(x) = x - 2\omega_i, T_5(x) = x + 2\omega_i.$$

Figure 1 indicates the spectral plane$[0,2\pi)^2$, for

$$\omega_i = \{0.75,1.5,2.25\}.$$

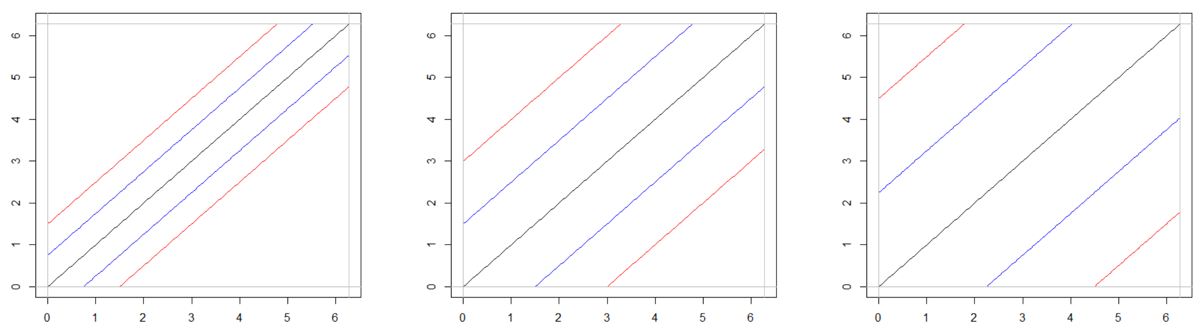

Figure 1: The spectral square of the process, Left: $\omega_i = 0.5$, Middle: $\omega_i = 1$, and Right: $\omega_i = 2$.

First, we estimated the Type I error probability $(\hat{\alpha})$ and power $(\hat{\pi})$ based on 1000 replications and 1000 iterations. Then we graph Q–Q plot for the test statistic ${\chi^2}^*$based on the computed values of the simulation runs.

Table1 report the values of $\hat{\alpha}$ (in rows: 1th, 14th and 27th) and $\hat{\pi}$ (other rows). The results indicates that the values of $\hat{\alpha}$ is very close to the considered size ($\alpha = 0.05$), especially for large values of (n_1, n_2, n_3). Also the power studies show that the proposed method excellently discriminate H_0 from H_1.

Table 1: The values of $\hat{\alpha}$ and $\hat{\pi}$ for the introduced approach

			(n_1, n_2, n_3)			
ω_1	ω_2	ω_3	$(100, 50, 75)$	$(150, 75, 100)$	$(200, 150, 100)$	$(500, 250, 300)$
0.75	0.75	0.75	0.052	0.050	0.050	0.048
		1.5	0.796	0.885	0.924	0.985
		2.25	0.757	0.878	0.942	0.990
	1.5	0.75	0.777	0.883	0.927	0.971
		1.5	0.770	0.875	0.909	0.976
		2.25	0.775	0.889	0.928	0.976
	2.25	0.75	0.768	0.883	0.914	0.994
		1.5	0.755	0.857	0.955	0.996
		2.25	0.752	0.860	0.903	0.976
1.5	0.75	0.75	0.783	0.854	0.919	0.983
		1.5	0.796	0.876	0.943	0.985
		2.25	0.774	0.893	0.934	0.995
	1.5	0.75	0.758	0.879	0.915	0.998
		1.5	0.052	0.051	0.049	0.048
		2.25	0.790	0.874	0.946	0.993
	2.25	0.75	0.788	0.870	0.929	0.998
		1.5	0.779	0.872	0.956	0.996
		2.25	0.750	0.877	0.922	0.987
2.25	0.75	0.75	0.798	0.896	0.912	0.991
		1.5	0.769	0.874	0.934	0.972
		2	0.767	0.876	0.932	0.984
	1.5	0.75	0.756	0.863	0.933	0.986
		1.5	0.791	0.876	0.909	0.971
		2.25	0.754	0.887	0.920	0.990
	2.25	0.75	0.754	0.874	0.905	0.992
		1.5	0.768	0.880	0.919	0.995
		0.051	0.051	0.049	0.049	0.051

5. Real Data

This section is devoted to illustrate the ability of introduced approach in practical cases. The dataset includes the first difference of centered moving average filter 2×12 moving average (MA) applied for logarithm of industrial production index (IPI) in Poland (2005 = 100%) since January 1995 untile December 2009, Lenart and Pipien (2013b). We split this dataset in three parts with equal sizes. The spectral frequency squares of these parts are given in Figure 2. The results detect ACS time series with spectra on the lines $T_j(x) = x \pm \alpha, \alpha \in \{0.062, 0.153, 0.258\}$. This result verifies the given result in Lenart and Pipien (2013b). Then the introduced technique is used to test the the hypothesis $\boldsymbol{\Sigma}^{(1)} = \boldsymbol{\Sigma}^{(2)} = \boldsymbol{\Sigma}^{(3)}$ (or equivalently, $\boldsymbol{f_1} = \boldsymbol{f_2} = \boldsymbol{f_3}$). Table 2 summarizes the results. As can be seen, since the p value is more than 0.05, thus the null hypothesis can not be rejected and consequently the stochastic behaviours of all processes are similar.

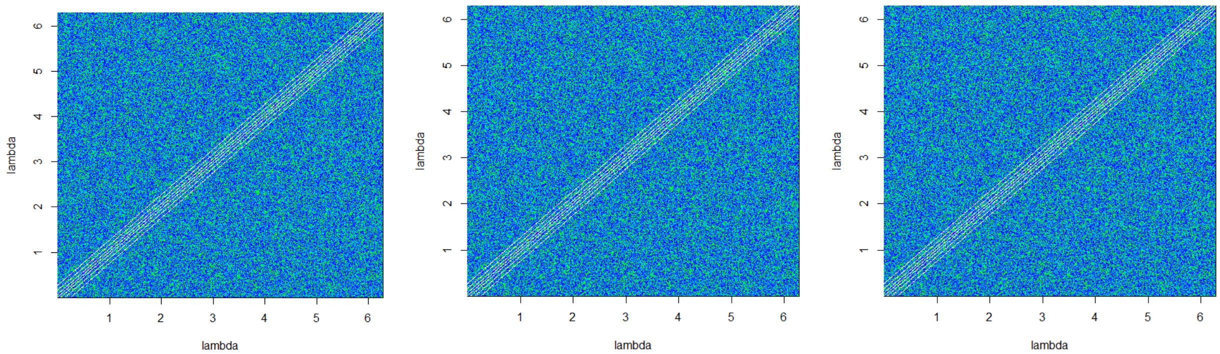

Figure 2: Spectral frequency square (Left: Part 1, Middle: Part 2, Right: Part 3)

Table 2: Testing the equality of different parts

Test Statistic	**P-Value**
$\chi_0^2 = 35.162$	0.763

Conflict of Interest

The authors declare that they have no conflict of interest.

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