The Reimann Hypothesis

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Abstract. In mathematics, the Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. Many consider it to be the most important unsolved problem in pure mathematics. It is one of the seven Millennium Prize Problems selected by the Clay Mathematics Institute to carry a US 1,000,000 prize for the first correct solution. In 1915, Ramanujan proved that under the assumption of the Riemann Hypothesis, the inequality $\sigma(n) < e^{\gamma} \times n \times \log\log n$ holds for all sufficiently large n, where $\sigma(n)$ is the sum-of-divisors function and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. In 1984, Guy Robin proved that the inequality is true for all $n > 5040$ if and only if the Riemann Hypothesis is true. In 2002, Lagarias proved that if the inequality $\sigma(n) \leq H_n + exp(H_n) \times \log H_n$ holds for all $n \geq 1$, then the Riemann Hypothesis is true, where H_n is the n^{th} harmonic number. In this work, we show certain properties of these both inequalities.

1. Introduction

As usual $\sigma(n)$ is the sum-of-divisors function of n [Cho+07]:

$$\sum_{d|n} d.$$

Define $f(n)$ to be $\frac{\sigma(n)}{n}$. Say $\mathsf{Robins}(n)$ holds provided

$$f(n) < e^{\gamma} \times \log\log n.$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, and log is the natural logarithm. Let H_n be $\sum_{j=1}^{n} \frac{1}{j}$. Say $\mathsf{Lagarias}(n)$ holds provided

$$\sigma(n) \leq H_n + exp(H_n) \times \log H_n.$$

The importance of this property is:

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Theorem 1.1. [RH] *If* $\mathsf{Robins}(n)$ *holds for all* $n > 5040$*, then the Riemann Hypothesis is true [Lag02]. If* $\mathsf{Lagarias}(n)$ *holds for all* $n \geq 1$*, then the Riemann Hypothesis is true [Lag02].*

It is known that $\mathsf{Robins}(n)$ and $\mathsf{Lagarias}(n)$ hold for many classes of numbers n. We known this:

Lemma 1.2. [known] *If* $\mathsf{Robins}(n)$ *holds for some* $n > 5040$*, then* $\mathsf{Lagarias}(n)$ *holds [Lag02].*

We recall that an integer n is said to be square free if for every prime divisor q of n we have $q^2 \nmid n$ [Cho+07]. $\mathsf{Robins}(n)$ holds for all $n > 5040$ that are square free [Cho+07]. Let $\mathsf{core}(n)$ denotes the square free kernel of a natural number n [Cho+07]. We can show this:

Theorem 1.3. [pi] *Let* $\frac{\pi^2}{6} \times \log\log \mathsf{core}(n) \leq \log\log n$ *for some* $n > 5040$*. Then* $\mathsf{Robins}(n)$ *holds.*

Moreover, we finally prove these theorems:

Theorem 1.4. [1-main] $\mathsf{Robins}(n)$ *holds for all* $n > 5040$ *when* $q_m \nmid n$ *for* $q_m \leq 113$*.*

Theorem 1.5. [2-main] *Let* $n > 5040$ *and* $n = r \times q$*, where* q *denotes the largest prime factor of* n *and* q *is a sufficiently large number. If* $\mathsf{Robins}(r)$ *holds, then* $\mathsf{Lagarias}(n)$ *holds.*

2. Known Results

We use that the following are known:

Lemma 2.1. [sigma-formula]

$$\sigma(n) = \prod_{p^k \| n} \frac{p^{k+1} - 1}{p - 1} \qquad \text{[Cho+07]}$$

Lemma 2.2. [sigma-bound]

$$f(n) < \prod_{p \mid n} \frac{p}{p-1}. \qquad \text{[Cho+07]}$$

Lemma 2.3. [zeta]

$$\prod_{k=1}^{\infty} \frac{1}{1 - \frac{1}{q_k^2}} = \zeta(2) = \frac{\pi^2}{6}. \qquad \text{[Edw01]}$$

Lemma 2.4. [log-bound]

$$H_n > \log n + \gamma = \log(e^{\gamma} \times n). \qquad \text{[Lag02]}$$

Lemma 2.5. [harmonic-bound]

$$\prod_{p \le n} \frac{p}{p-1} < e^{\gamma} \times H_n. \qquad \text{[RS62]}$$

Lemma 2.6. [down-bound] *For $x \ge 286$,*

$$\prod_{p \le x} \frac{p}{p-1} < e^{\gamma} \times (\log x + \frac{1}{2 \times \log x}). \qquad \text{[RS62]}$$

3. A Central Lemma

The following is a key lemma. It gives an upper bound on $f(n)$ that holds for all n. The bound is too weak to prove $\mathsf{Robins}(n)$ directly, but is critical because it holds for all n. Further the bound only uses the primes that divide n and not how many times they divide n. This is a key insight.

Lemma 3.1. [pro] *Let $n > 1$ and let all its prime divisors be $q_1 < \cdots < q_m$. Then,*

$$f(n) < \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i}.$$

Proof. We use that lemma 2.2 [sigma-bound]:

$$f(n) < \prod_{i=1}^{m} \frac{q_i}{q_i - 1}.$$

Now for $q > 1$,

$$\frac{1}{1 - \frac{1}{q^2}} = \frac{q^2}{q^2 - 1}.$$

So

$$\begin{aligned} \frac{1}{1 - \frac{1}{q^2}} \times \frac{q+1}{q} &= \frac{q^2}{q^2 - 1} \times \frac{q+1}{q} \\ &= \frac{q}{q-1}. \end{aligned}$$

Then by lemma 2.3 [zeta],

$$\prod_{k=1}^{m} \frac{1}{1 - \frac{1}{q_k^2}} < \zeta(2) = \frac{\pi^2}{6}.$$

Putting this together yields the proof:

$$\begin{aligned} f(n) &< \prod_{i=1}^{m} \frac{q_i}{q_i - 1} \\ &\leq \prod_{i=1}^{m} \frac{1}{1 - \frac{1}{q_i^2}} \times \frac{q_i + 1}{q_i} \\ &< \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i}. \end{aligned}$$

□

4. A Condition on $\mathsf{core}(n)$

4.1. **A Particular Case.** We prove the Robin's inequality for this particular case:

Lemma 4.1. [case] $\mathsf{Robins}(n)$ *holds for all* $n > 5040$ *when* $\mathsf{core}(n) \in \{2, 3, 5, 6, 10, 14, 15, 21, 30, 35, 42, 70, 105, 210\}$.

Proof. Let $n > 5040$. Specifically, let $\mathsf{core}(n)$ be the product of the primes $q_1, \ldots, q_m$, such that $\{q_1, \ldots, q_m\} \subseteq \{2, 3, 5\}$. We need to prove that

$$f(n) < e^{\gamma} \times \log \log n$$

that is true when

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} \leq e^{\gamma} \times \log \log n$$

is also true, because of lemma 2.2 [sigma-bound]. Then, we have that

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} \leq \frac{2 \times 3 \times 5}{1 \times 2 \times 4} = 3.75 < e^{\gamma} \times \log \log(5040) \approx 3.81.$$

However, for $n > 5040$

$$e^{\gamma} \times \log \log(5040) < e^{\gamma} \times \log \log n$$

and hence, the proof is completed for that case. Hence, we only need to prove the Robin's inequality is true for every natural number $n = 2^{a_1} \times 3^{a_2} \times 5^{a_3} \times 7^{a_4} > 5040$ such that $a_1, a_2, a_3 \geq 0$ and $a_4 \geq 1$ are integers. In addition, we know the Robin's inequality is true for every natural number $n > 5040$ such that $7^k \mid n$ and $7^7 \nmid n$ for some integer $1 \leq k \leq 6$ [Her18]. Therefore, we need to prove this case for those natural numbers $n > 5040$ such that $7^7 \mid n$. In this way, we have

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} \leq \frac{2 \times 3 \times 5 \times 7}{1 \times 2 \times 4 \times 6} = 4.375 < e^{\gamma} \times \log \log(7^7) \approx 4.65.$$

However, we know for $n > 5040$ and $7^7 \mid n$ such that

$$e^{\gamma} \times \log\log(7^7) \leq e^{\gamma} \times \log\log n$$

and as a consequence, the proof is completed. $\square$

4.2. **Main Insight.** The next theorem is a main insight. It extends the class of n so that $\mathsf{Robins}(n)$ holds. The key is that the class on n depend on how close n is to $\mathsf{core}(n)$. The usual classes of such n are defined by their prime structure not by an inequality. This is perhaps one of the main insights.

Theorem 4.2. *Let* $\frac{\pi^2}{6} \times \log\log \mathsf{core}(n) \leq \log\log n$ *for some* $n > 5040$. *Then* $\mathsf{Robins}(n)$ *holds.*

Proof. Let $n' = \mathsf{core}(n)$. Let n' be the product of the distinct primes $q_1, \ldots, q_m$. By assumption we have that

$$\frac{\pi^2}{6} \times \log\log n' \leq \log\log n.$$

When $n' \leq 5040$, $\mathsf{Robins}(n')$ holds if $n' \notin \{2, 3, 5, 6, 10, 30\}$ [Cho+07]. However, we can ignore this case, since $\mathsf{Robins}(n)$ holds for all $n > 5040$ when $\mathsf{core}(n) \in \{2, 3, 5, 6, 10, 30\}$ because of lemma 4.1 [case]. When $n' > 5040$, we know that $\mathsf{Robins}(n')$ holds and so

$$f(n') < e^{\gamma} \times \log\log n'.$$

By previous lemma 3.1 [pro]

$$f(n) < \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i}.$$

Suppose by way of contradiction that $\mathsf{Robins}(n)$ fails. Then

$$f(n) \geq e^{\gamma} \times \log\log n.$$

We claim that

$$\frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i} > e^{\gamma} \times \log\log n.$$

Since otherwise we would have a contradiction. This shows that

$$\frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i} > \frac{\pi^2}{6} \times e^{\gamma} \times \log\log n'.$$

Thus

$$\prod_{i=1}^{m} \frac{q_i + 1}{q_i} > e^{\gamma} \times \log\log n',$$

and

$$\prod_{i=1}^{m} \frac{q_i + 1}{q_i} > f(n'),$$

This is a contradiction since $f(n')$ is equal to

$$\frac{(q_1 + 1) \times \cdots \times (q_m + 1)}{q_1 \times \cdots \times q_m}.$$

□

5. On Possible Counterexamples

Lemma 5.1. [counter] *Let $n > 5040$ and $n = r \times q$, where q denotes the largest prime factor of n. We have that $q < \log n$, when* $\mathsf{Robins}(r)$ *holds, but* $\mathsf{Robins}(n)$ *does not.*

Proof. So assume that $q \geq \log n$. This implies that $q \times \log q \geq (\log n) \times \log \log n > (\log n) \times \log \log r$ and hence

$$\frac{q}{\log n} > \frac{\log \log r}{\log q}.$$

This implies that

$$\frac{q \times (\log \log n - \log \log r)}{\log q} > \frac{\log \log r}{\log q},$$

where we used that

$$\frac{\log \log n - \log \log r}{\log q} = \frac{1}{\log n - \log r} \int_{\log r}^{\log n} \frac{dt}{t} > \frac{1}{\log n}. \qquad \text{[Cho+07]}$$

This inequality is equivalent with $(1 + \frac{1}{q}) \times \log \log r < \log \log n$. Now we infer that

$$\frac{\sigma(n)}{n} = \frac{\sigma(q \times r)}{q \times r} \leq (1+\frac{1}{q}) \times \frac{\sigma(r)}{r} < (1+\frac{1}{q}) \times e^{\gamma} \times \log \log r < e^{\gamma} \times \log \log n$$

because of we know that $\mathsf{Robins}(r)$ holds and where we used that σ is submultiplicative (that is $\sigma(q \times r) \leq \sigma(q) \times \sigma(r)$) [Cho+07]. The last inequality contradicts our assumption that $\mathsf{Robins}(n)$ does not hold. □

6. Robin's Divisibility

Lemma 6.1. [up-bound] *For $x \geq 11$, we have*

$$\sum_{q \leq x} \frac{1}{q} < \log \log x + \gamma - 0.12$$

where $q \leq x$ means all the primes lesser than or equal to x.

Proof. For $x > 1$, we have

$$\sum_{q \leq x} \frac{1}{q} < \log \log x + B + \frac{1}{\log^2 x}$$

where

$$B = 0.2614972128 \cdots$$

is the (Meissel-)Mertens constant, since this is a proven result from the article reference [RS62]. This is the same as

$$\sum_{q \leq x} \frac{1}{q} < \log \log x + \gamma - (C - \frac{1}{\log^2 x})$$

where $\gamma - B = C > 0.31$, because of $\gamma > B$. If we analyze $(C - \frac{1}{\log^2 x})$, then this complies with

$$(C - \frac{1}{\log^2 x}) > (0.31 - \frac{1}{\log^2 11}) > 0.12$$

for $x \geq 11$ and thus, we finally prove

$$\sum_{q \leq x} \frac{1}{q} < \log \log x + \gamma - (C - \frac{1}{\log^2 x}) < \log \log x + \gamma - 0.12.$$

□

Theorem 6.2. [strict] *Given a square free number*

$$n = q_1 \times \cdots \times q_m$$

such that $q_1, q_2, \cdots, q_m$ *are odd prime numbers, the greatest prime divisor of* n *is greater than* 7 *and* $3 \nmid n$*, then we obtain the following inequality*

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \sigma(n) \leq e^{\gamma} \times n \times \log \log(2^{19} \times n).$$

Proof. This proof is very similar with the demonstration in theorem 1.1 from the article reference [Cho+07]. By induction with respect to $\omega(n)$, that is the number of distinct prime factors of n [Cho+07]. Put $\omega(n) = m$ [Cho+07]. We need to prove the assertion for those integers with $m = 1$. From a square free number n, we obtain

$$\sigma(n) = (q_1 + 1) \times (q_2 + 1) \times \cdots \times (q_m + 1) [\text{eq} : 1] \qquad 6.1$$

when $n = q_1 \times q_2 \times \cdots \times q_m$ [Cho+07]. In this way, for every prime number $q_i \geq 11$, then we need to prove

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (1 + \frac{1}{q_i}) \leq e^{\gamma} \times \log \log(2^{19} \times q_i). [\text{eq} : 2] \qquad 6.2$$

For $q_i = 11$, we have

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (1 + \frac{1}{11}) \leq e^{\gamma} \times \log\log(2^{19} \times 11)$$

is actually true. For another prime number $q_i > 11$, we have

$$(1 + \frac{1}{q_i}) < (1 + \frac{1}{11})$$

and

$$\log\log(2^{19} \times 11) < \log\log(2^{19} \times q_i)$$

which clearly implies that the inequality 6.2 is true for every prime number $q_i \geq 11$. Now, suppose it is true for $m - 1$, with $m \geq 2$ and let us consider the assertion for those square free n with $\omega(n) = m$ [Cho+07]. So let $n = q_1 \times \cdots \times q_m$ be a square free number and assume that $q_1 < \cdots < q_m$ for $q_m \geq 11$.

Case 1: $q_m \geq \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = \log(2^{19} \times n)$.

By the induction hypothesis we have

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (q_1 + 1) \times \cdots \times (q_{m-1} + 1) \leq e^{\gamma} \times q_1 \times \cdots \times q_{m-1} \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1})$$

and hence

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (q_1 + 1) \times \cdots \times (q_{m-1} + 1) \times (q_m + 1) \leq$$

$$e^{\gamma} \times q_1 \times \cdots \times q_{m-1} \times (q_m + 1) \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1})$$

when we multiply the both sides of the inequality by $(q_m + 1)$. We want to show

$$e^{\gamma} \times q_1 \times \cdots \times q_{m-1} \times (q_m + 1) \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1}) \leq$$

$$e^{\gamma} \times q_1 \times \cdots \times q_{m-1} \times q_m \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = e^{\gamma} \times n \times \log\log(2^{19} \times n).$$

Indeed the previous inequality is equivalent with

$$q_m \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) \geq (q_m + 1) \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1})$$

or alternatively

$$\frac{q_m \times (\log\log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1}))}{\log q_m} \geq$$

$$\frac{\log\log(2^{19} \times q_1 \times \cdots \times q_{m-1})}{\log q_m}.$$

From the reference [Cho+07], we have if $0 < a < b$, then

$$\frac{\log b - \log a}{b - a} = \frac{1}{(b - a)} \int_a^b \frac{dt}{t} > \frac{1}{b}. \text{[eq : 3]} \qquad 6.3$$

We can apply the inequality 6.3 to the previous one just using $b = \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m)$ and $a = \log(2^{19} \times q_1 \times \cdots \times q_{m-1})$. Certainly, we have

$$\log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \log(2^{19} \times q_1 \times \cdots \times q_{m-1}) =$$

$$\log \frac{2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m}{2^{19} \times q_1 \times \cdots \times q_{m-1}} = \log q_m.$$

In this way, we obtain

$$\frac{q_m \times (\log\log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1}))}{\log q_m} >$$

$$\frac{q_m}{\log(2^{19} \times q_1 \times \cdots \times q_m)}.$$

Using this result we infer that the original inequality is certainly satisfied if the next inequality is satisfied

$$\frac{q_m}{\log(2^{19} \times q_1 \times \cdots \times q_m)} \geq \frac{\log\log(2^{19} \times q_1 \times \cdots \times q_{m-1})}{\log q_m}$$

which is trivially true for $q_m \geq \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m)$ [Cho+07].

Case 2: $q_m < \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = \log(2^{19} \times n)$.

We need to prove

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \frac{\sigma(n)}{n} \leq e^{\gamma} \times \log\log(2^{19} \times n).$$

We know $\frac{3}{2} < 1.503 < \frac{4}{2.66}$. Nevertheless, we could have

$$\frac{3}{2} \times \frac{\sigma(n)}{n} \times \frac{\pi^2}{6} < \frac{4 \times \sigma(n)}{3 \times n} \times \frac{\pi^2}{2 \times 2.66}$$

and therefore, we only need to prove

$$\frac{\sigma(3 \times n)}{3 \times n} \times \frac{\pi^2}{5.32} \leq e^{\gamma} \times \log\log(2^{19} \times n)$$

where this is possible because of $3 \nmid n$. If we apply the logarithm to the both sides of the inequality, then we obtain

$$\log(\frac{\pi^2}{5.32}) + (\log(3+1) - \log 3) + \sum_{i=1}^{m} (\log(q_i+1) - \log q_i) \leq \gamma + \log\log\log(2^{19} \times n).$$

From the reference [Cho+07], we note

$$\log(q_1 + 1) - \log q_1 = \int_{q_1}^{q_1+1} \frac{dt}{t} < \frac{1}{q_1}.$$

In addition, note $\log(\frac{\pi^2}{5.32}) < \frac{1}{2} + 0.12$. However, we know

$$\gamma + \log\log q_m < \gamma + \log\log\log(2^{19} \times n)$$

since $q_m < \log(2^{19} \times n)$ and therefore, it is enough to prove

$$0.12 + \frac{1}{2} + \frac{1}{3} + \frac{1}{q_1} + \cdots + \frac{1}{q_m} \leq 0.12 + \sum_{q \leq q_m} \frac{1}{q} \leq \gamma + \log\log q_m$$

where $q_m \geq 11$. In this way, we only need to prove

$$\sum_{q \leq q_m} \frac{1}{q} \leq \gamma + \log\log q_m - 0.12$$

which is true according to the lemma 6.1 [up-bound] when $q_m \geq 11$. In this way, we finally show the theorem is indeed satisfied. □

Theorem 6.3. [btw2-3] Robins(n) *holds for all* $n > 5040$ *when* $3 \nmid n$. *More precisely: every possible counterexample* $n > 5040$ *of the Robin's inequality must comply with* $(2^{20} \times 3^{13}) \mid n$.

Proof. We will check the Robin's inequality is true for every natural number $n = q_1^{a_1} \times q_2^{a_2} \times \cdots \times q_m^{a_m} > 5040$ such that $q_1, q_2, \cdots, q_m$ are prime numbers, $a_1, a_2, \cdots, a_m$ are natural numbers and $3 \nmid n$. We know this is true when the greatest prime divisor of $n > 5040$ is lesser than or equal to 7 according to the lemma 4.1 [case]. Therefore, the remaining case is when the greatest prime divisor of $n > 5040$ is greater than 7. We need to prove

$$\frac{\sigma(n)}{n} < e^{\gamma} \times \log\log n$$

that is true when

$$\frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i} \leq e^{\gamma} \times \log\log n$$

according to the lemma 3.1 [pro]. Using the equation 6.1, we obtain that will be equivalent to

$$\frac{\pi^2}{6} \times \frac{\sigma(n')}{n'} \leq e^{\gamma} \times \log\log n$$

where $n' = q_1 \times \cdots \times q_m$ is the core(n) [Cho+07]. However, the Robin's inequality has been proved for all integers n not divisible by 2 (which are bigger than 10) [Cho+07]. Hence, we only need to prove the Robin's inequality is true when $2 \mid n'$. In addition, we know the Robin's inequality is true for every natural number $n > 5040$ such that $2^k \mid n$ and $2^{20} \nmid n$ for some integer $1 \leq k \leq 19$ [Her18]. Consequently, we only need to prove the Robin's inequality is true for all $n > 5040$ such that $2^{20} \mid n$ and thus,

$$e^{\gamma} \times n' \times \log\log(2^{19} \times \frac{n'}{2}) < e^{\gamma} \times n' \times \log\log n$$

because of $2^{19} \times \frac{n'}{2} < n$ when $2^{20} \mid n$ and $2 \mid n'$. In this way, we only need to prove

$$\frac{\pi^2}{6} \times \sigma(n') \leq e^{\gamma} \times n' \times \log\log(2^{19} \times \frac{n'}{2}).$$

According to the equation 6.1 and $2 \mid n'$, we have

$$\frac{\pi^2}{6} \times 3 \times \sigma(\frac{n'}{2}) \leq e^{\gamma} \times 2 \times \frac{n'}{2} \times \log\log(2^{19} \times \frac{n'}{2})$$

which is the same as

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \sigma(\frac{n'}{2}) \leq e^{\gamma} \times \frac{n'}{2} \times \log\log(2^{19} \times \frac{n'}{2})$$

that is true according to the theorem 6.2 [strict] when $3 \nmid \frac{n'}{2}$. In addition, we know the Robin's inequality is true for every natural number $n > 5040$ such that $3^k \mid n$ and $3^{13} \nmid n$ for some integer $1 \leq k \leq 12$ [Her18]. Consequently, we only need to prove the Robin's inequality is true for all $n > 5040$ such that $2^{20} \mid n$ and $3^{13} \mid n$. To sum up, the proof is completed. □

Theorem 6.4. [btw5-7] $\mathsf{Robins}(n)$ *holds for all* $n > 5040$ *when* $5 \nmid n$ *or* $7 \nmid n$.

Proof. We need to prove

$$f(n) < e^{\gamma} \times \log\log n$$

when $(2^{20} \times 3^{13}) \mid n$. Suppose that $n = 2^a \times 3^b \times m$, where $a \geq 20$, $b \geq 13$, $2 \nmid m$, $3 \nmid m$ and $5 \nmid m$ or $7 \nmid m$. Therefore, we need to prove

$$f(2^a \times 3^b \times m) < e^{\gamma} \times \log\log(2^a \times 3^b \times m).$$

We know

$$f(2^a \times 3^b \times m) = f(3^b) \times f(2^a \times m)$$

since s is multiplicative [Voj20]. In addition, we know $f(3^b) < \frac{3}{2}$ for every natural number b [Voj20]. In this way, we have

$$f(3^b) \times f(2^a \times m) < \frac{3}{2} \times f(2^a \times m).$$

Now, consider

$$\frac{3}{2} \times f(2^a \times m) = \frac{9}{8} \times f(3) \times f(2^a \times m) = \frac{9}{8} \times f(2^a \times 3 \times m)$$

where $f(3) = \frac{4}{3}$ since s is multiplicative [Voj20]. Nevertheless, we have

$$\frac{9}{8} \times f(2^a \times 3 \times m) < f(5) \times f(2^a \times 3 \times m) = f(2^a \times 3 \times 5 \times m)$$

and

$$\frac{9}{8} \times f(2^a \times 3 \times m) < f(7) \times f(2^a \times 3 \times m) = f(2^a \times 3 \times 7 \times m)$$

where $5 \nmid m$ or $7 \nmid m$, $f(5) = \frac{6}{5}$ and $f(7) = \frac{8}{7}$. However, we know the Robin's inequality is true for $2^a \times 3 \times 5 \times m$ and $2^a \times 3 \times 7 \times m$ when $a \geq 20$, since this is true for every natural number $n > 5040$ such that $3^k \mid n$ and $3^{13} \nmid n$ for some integer $1 \leq k \leq 12$ [Her18]. Hence, we would have

$$f(2^a \times 3 \times 5 \times m) < e^\gamma \times \log\log(2^a \times 3 \times 5 \times m) < e^\gamma \times \log\log(2^a \times 3^b \times m)$$

and

$$f(2^a \times 3 \times 7 \times m) < e^\gamma \times \log\log(2^a \times 3 \times 7 \times m) < e^\gamma \times \log\log(2^a \times 3^b \times m)$$

when $b \geq 13$. □

Theorem 6.5. [btw11-47] Robins(n) *holds for all* $n > 5040$ *when* $q_m \nmid n$ *for* $11 \leq q_m \leq 47$.

Proof. We know the Robin's inequality is true for every natural number $n > 5040$ such that $7^k \mid n$ and $7^7 \nmid n$ for some integer $1 \leq k \leq 6$ [Her18]. We need to prove

$$f(n) < e^\gamma \times \log\log n$$

when $(2^{20} \times 3^{13} \times 7^7) \mid n$. Suppose that $n = 2^a \times 3^b \times 7^c \times m$, where $a \geq 20$, $b \geq 13$, $c \geq 7$, $2 \nmid m$, $3 \nmid m$, $7 \nmid m$, $q_m \nmid m$ and $11 \leq q_m \leq 47$. Therefore, we need to prove

$$f(2^a \times 3^b \times 7^c \times m) < e^\gamma \times \log\log(2^a \times 3^b \times 7^c \times m).$$

We know

$$f(2^a \times 3^b \times 7^c \times m) = f(7^c) \times f(2^a \times 3^b \times m)$$

since s is multiplicative [Voj20]. In addition, we know $f(7^c) < \frac{7}{6}$ for every natural number c [Voj20]. In this way, we have

$$f(7^c) \times f(2^a \times 3^b \times m) < \frac{7}{6} \times f(2^a \times 3^b \times m).$$

However, that would be equivalent to

$$\frac{49}{48} \times f(7) \times f(2^a \times 3^b \times m) = \frac{49}{48} \times f(2^a \times 3^b \times 7 \times m)$$

where $f(7) = \frac{8}{7}$. In addition, we know

$$\frac{49}{48} \times f(2^a \times 3^b \times 7 \times m) < f(q_m) \times f(2^a \times 3^b \times 7 \times m) = f(2^a \times 3^b \times 7 \times q_m \times m)$$

where $q_m \nmid m$, $f(q_m) = \frac{q_m+1}{q_m}$ and $11 \leq q_m \leq 47$. Nevertheless, we know the Robin's inequality is true for $2^a \times 3^b \times 7 \times q_m \times m$ when $a \geq 20$

and $b \geq 13$, since this is true for every natural number $n > 5040$ such that $7^k \mid n$ and $7^7 \nmid n$ for some integer $1 \leq k \leq 6$ [Her18]. Hence, we would have

$$f(2^a \times 3^b \times 7 \times q_m \times m) < e^{\gamma} \times \log\log(2^a \times 3^b \times 7 \times q_m \times m) < e^{\gamma} \times \log\log(2^a \times 3^b \times 7^c \times m)$$

when $c \geq 7$ and $11 \leq q_m \leq 47$. □

Theorem 6.6. [btw53-113] Robins(n) *holds for all* $n > 5040$ *when* $q_m \nmid n$ *for* $53 \leq q_m \leq 113$.

Proof. We know the Robin's inequality is true for every natural number $n > 5040$ such that $11^k \mid n$ and $11^6 \nmid n$ for some integer $1 \leq k \leq 5$ [Her18]. We need to prove

$$f(n) < e^{\gamma} \times \log\log n$$

when $(2^{20} \times 3^{13} \times 11^6) \mid n$. Suppose that $n = 2^a \times 3^b \times 11^c \times m$, where $a \geq 20$, $b \geq 13$, $c \geq 6$, $2 \nmid m$, $3 \nmid m$, $11 \nmid m$, $q_m \nmid m$ and $53 \leq q_m \leq 113$. Therefore, we need to prove

$$f(2^a \times 3^b \times 11^c \times m) < e^{\gamma} \times \log\log(2^a \times 3^b \times 11^c \times m).$$

We know

$$f(2^a \times 3^b \times 11^c \times m) = f(11^c) \times f(2^a \times 3^b \times m)$$

since s is multiplicative [Voj20]. In addition, we know $f(11^c) < \frac{11}{10}$ for every natural number c [Voj20]. In this way, we have

$$f(11^c) \times f(2^a \times 3^b \times m) < \frac{11}{10} \times f(2^a \times 3^b \times m).$$

However, that would be equivalent to

$$\frac{121}{120} \times f(11) \times f(2^a \times 3^b \times m) = \frac{121}{120} \times f(2^a \times 3^b \times 11 \times m)$$

where $f(11) = \frac{12}{11}$. In addition, we know

$$\frac{121}{120} \times f(2^a \times 3^b \times 11 \times m) < f(q_m) \times f(2^a \times 3^b \times 11 \times m) = f(2^a \times 3^b \times 11 \times q_m \times m)$$

where $q_m \nmid m$, $f(q_m) = \frac{q_m+1}{q_m}$ and $53 \leq q_m \leq 113$. Nevertheless, we know the Robin's inequality is true for $2^a \times 3^b \times 11 \times q_m \times m$ when $a \geq 20$ and $b \geq 13$, since this is true for every natural number $n > 5040$ such that $11^k \mid n$ and $11^6 \nmid n$ for some integer $1 \leq k \leq 5$ [Her18]. Hence, we would have

$$f(2^a \times 3^b \times 11 \times q_m \times m) < e^{\gamma} \times \log\log(2^a \times 3^b \times 11 \times q_m \times m) < e^{\gamma} \times \log\log(2^a \times 3^b \times 11^c \times m)$$

when $c \geq 6$ and $53 \leq q_m \leq 113$. □

7. Proof of Main Theorems

Theorem 7.1. $\mathsf{Robins}(n)$ *holds for all* $n > 5040$ *when* $q_m \nmid n$ *for* $q_m \leq 113$.

Proof. This is a compendium of the results from the Theorems 6.3 [btw2-3], 6.4 [btw5-7], 6.5 [btw11-47] and 6.6 [btw53-113]. ☐

Theorem 7.2. *Let* $n > 5040$ *and* $n = r \times q$, *where* q *denotes the largest prime factor of* n *and* q *is a sufficiently large number. If* $\mathsf{Robins}(r)$ *holds, then* $\mathsf{Lagarias}(n)$ *holds.*

Proof. We need to prove

$$\sigma(n) \leq H_n + exp(H_n) \times \log H_n.$$

We know if $\mathsf{Robins}(n)$ holds for $n > 5040$, then $\mathsf{Lagarias}(n)$ holds because of lemma 1.2 [known]. In addition, $\mathsf{Lagarias}(n)$ has been checked for all $n \leq 5040$. Now suppose that $\mathsf{Robins}(r)$ holds, but $\mathsf{Robins}(n)$ does not. Let's multiply by e^{γ} the both sides of inequality and thus,

$$e^{\gamma} \times \sigma(n) \leq e^{\gamma} \times H_n + e^{\gamma} \times exp(H_n) \times \log H_n.$$

If we apply the lemma 2.5 [harmonic-bound], then we obtain that

$$\prod_{p \mid n} \frac{p}{p-1} \leq \prod_{p \leq n} \frac{p}{p-1} < e^{\gamma} \times H_n.$$

Hence, we obtain that

$$e^{\gamma} \times \sigma(n) - \prod_{p \mid n} \frac{p}{p-1} \leq e^{\gamma} \times exp(H_n) \times \log H_n.$$

That would be equivalent to

$$\prod_{p \mid n} \frac{p}{p-1} \times (e^{\gamma} \times \sigma(n) \times \prod_{p \mid n} \frac{p-1}{p} - 1) \leq e^{\gamma} \times exp(H_n) \times \log H_n.$$

We know that

$$\sigma(n) = \prod_{p^k \| n} \frac{p^{k+1} - 1}{p - 1}$$

because of lemma 2.1 [sigma-formula] and therefore

$$\begin{aligned} \sigma(n) \times \prod_{p \mid n} \frac{p-1}{p} &= \prod_{p^k \| n} \frac{p^{k+1} - 1}{p} \\ &= \prod_{p^k \| n} (p^k - \frac{1}{p}) \\ &< n. \end{aligned}$$

In this way, we can see that

$$\prod_{p|n} \frac{p}{p-1} \times (e^{\gamma} \times n - 1) \leq e^{\gamma} \times exp(H_n) \times \log H_n.$$

If we apply the lemma 2.4 [log-bound] to the previous inequality, then we obtain that

$$\prod_{p|n} \frac{p}{p-1} \times (e^{\gamma} \times n - 1) \leq e^{\gamma} \times (e^{\gamma} \times n) \times \log\log(e^{\gamma} \times n).$$

If we use the lemma 2.6 [down-bound], then we have that

$$e^{\gamma} \times (\log q + \frac{1}{2 \times \log q}) \times (e^{\gamma} \times n - 1) \leq e^{\gamma} \times (e^{\gamma} \times n) \times \log\log(e^{\gamma} \times n)$$

where q is the largest prime factor of n and q is a sufficiently large number. In addition, if we introduce the lemma 5.1 [counter], then we have

$$\frac{\log(q \times e^{\frac{1}{2 \times \log q}})}{\log(q + \gamma)} \leq \frac{e^{\gamma} \times n}{e^{\gamma} \times n - 1}.$$

However, we know that

$$\lim_{q \to \infty} \frac{\log(q \times e^{\frac{1}{2 \times \log q}})}{\log(q + \gamma)} \leq 1 \leq \frac{e^{\gamma} \times n}{e^{\gamma} \times n - 1}$$

for enough large values of q and therefore, the proof is completed. □

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