Note for the Perfect Numbers

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Article

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Abstract: This paper tackles a longstanding problem in number theory: the infinitude of perfect numbers. A perfect number is defined as a positive integer whose sum of all its divisors is equal to twice the number itself. While Euclid's method provides a framework for constructing even perfect numbers using Mersenne primes, the infinitude of Mersenne primes remains an open question. If there are finitely many Mersenne primes, then there would also be a finite number of even perfect numbers. In this note, showing that there are finitely many Mersenne primes, we provide a partial answer by proving that is false the infinitude of even perfect numbers. The proof utilizes elementary techniques and relies on properties of the divisor sum function (sigma function) and the inherent structure of prime numbers.

Keywords: perfect numbers; Mersenne primes; prime numbers; divisor sum function

MSC: 11A41; 11A25

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1. Introduction

Prime numbers, the building blocks of integers, have fascinated mathematicians for centuries. Their enigmatic distribution and seemingly random occurrence have fueled the quest to understand their nature. Within this realm lies a special subset known as Mersenne primes, giants in the prime number kingdom, named after Marin Mersenne, a 17th-century French mathematician. Mersenne primes are a particular breed - they are prime numbers that can be expressed in a very specific form: 2 raised to an exponent (n) minus 1 ($2^n - 1$). For example, 3 ($2^2 - 1$) and 7 ($2^3 - 1$) are both Mersenne primes. While this formula seems simple, the resulting prime numbers can be colossal. Unlike many prime number searches, which rely on complex algorithms, checking for Mersenne primes can be done with a relatively simple formula. This has led to the rise of distributed computing projects like the Great Internet Mersenne Prime Search (GIMPS), where volunteers contribute their computers' processing power to the hunt for these elusive giants.

The concept of perfect numbers has captivated mathematicians for millennia. Defined as positive integers where the sum of their divisors equals twice the number itself, these integers hold a unique charm. Euclid, the venerable Greek mathematician, established a method to construct even perfect numbers using Mersenne primes. This discovery sparked a centuries-long pursuit: are there infinitely many perfect numbers? For mathematicians, the answer was not so intuitive. Indeed, the absence of a definitive proof left a lingering question since the 3rd century BC. Rene Descartes, the 17th-century philosopher and mathematician, and Leonhard Euler, another mathematical giant, built upon this question and established crucial properties that perfect numbers must possess. Despite these efforts, the question remained unanswered.

This paper aims to unveil the long-sought answer. Whether there are infinitely many Mersenne primes or not still remains as an open question [1]. The Lenstra-Pomerance-Wagstaff conjecture claims that there are infinitely many Mersenne primes and predicts their order of growth and frequency [1]. By employing the concept of the divisor sum function (sigma function) and delving into the properties of prime numbers, we will demonstrate a crucial contradiction under the assumption that there are infinitely many Mersenne primes. This contradiction will definitively prove the existence of only a finite number of even perfect numbers.

2. Materials and methods

The divisor sum function, denoted by $\sigma(n)$, is an arithmetic function in number theory. It's essentially a way to represent the sum of all the positive divisors of a positive integer n. Define $f(n)$ as $\frac{\sigma(n)}{n}$. The multiplicity is an important property of this previous function.

Proposition 1. *Let $\prod_{i=1}^{r} q_i^{a_i}$ be the representation of n as a product of prime numbers $q_1 < \ldots < q_r$ with natural numbers $a_1, \ldots, a_r$ as exponents. Then [2, Lemma 1 pp. 2],*

$$\begin{aligned} f(n) &= \left(\prod_{i=1}^{r} \frac{q_i}{q_i - 1}\right) \cdot \prod_{i=1}^{r}\left(1 - \frac{1}{q_i^{a_i+1}}\right) \\ &= \prod_{i=1}^{r} \frac{q_i^{a_i+1} - 1}{q_i^{a_i} \cdot (q_i - 1)} \\ &= \prod_{i=1}^{r} f(q_i^{a_i}). \end{aligned}$$

Leonhard Euler studied the following value of the Riemann zeta function (1734) [3].

Proposition 2. *We define [3, (1) pp. 1070]:*

$$\zeta(2) = \prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1} = \frac{\pi^2}{6},$$

where q_k is the kth prime number. By definition, we have

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

where n denotes a natural number. Leonhard Euler proved in his solution to the Basel problem that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1} = \frac{\pi^2}{6},$$

where $\pi \approx 3.14159$ is a well-known constant linked to several areas in mathematics such as number theory, geometry, etc.

We can easily prove the value of the following constant

$$\Upsilon = \sum_{n=0}^{\infty} \frac{1}{2^n} = 2$$

using geometric series [4].

In mathematics, a Mersenne prime is a prime number that is one less than a power of two. That is, it is a prime number of the form $M_n = 2^n - 1$ for some integer n. If n is a composite number then so is $2^n - 1$. Therefore, an equivalent definition of the Mersenne primes is that they are the prime numbers of the form $M_q = 2^q - 1$ for some prime q. Numbers of the form $M_n = 2^n - 1$ without the primality requirement may be called Mersenne numbers. In about 300 BC Euclid showed that if $2^q - 1$ is prime then $2^{q-1} \cdot (2^q - 1)$ is perfect. Later, Leonhard Euler proved that all the even perfect numbers are in the form that Euclid showed. This is known as the Euclid-Euler theorem.

Putting all together yields a proof that there are finitely many Mersenne primes.

3. Results

This is the main theorem.

Theorem 1. *There are finitely many Mersenne primes.*

Proof. Suppose that there are infinitely many Mersenne primes. Consider the sequence PM_n of prime numbers such that PM_k is the kth Mersenne prime. Consider also the geometric series

$$\mathrm{Y} = \sum_{n=0}^{\infty} \frac{1}{2^n} = 2.$$

Now, let's take the first Mersenne prime $PM_1 = 3$. We can express the constant Y as:

$$\mathrm{Y} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

Multiplying it by $\frac{1}{PM_1+1} = \frac{1}{4}$, we obtain:

$$\frac{1}{4} \cdot \mathrm{Y} = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \dots.$$

Next, we perform the following subtraction:

$$\begin{aligned} \mathrm{Y} - \frac{1}{4} \cdot \mathrm{Y} &= \left(1 - \frac{1}{4}\right) \cdot \mathrm{Y} \\ &= \frac{3}{4} \cdot \mathrm{Y} \\ &= \frac{1}{f(PM_1)} \cdot \mathrm{Y} \\ &= 1 + \frac{1}{2}. \end{aligned}$$

Here, we use the fact that $f(PM_1) = f(3) = \frac{4}{3}$ by Proposition 1 (which defines the function f). Furthermore, we know from geometric series properties that:

$$\begin{aligned} 1 + \frac{1}{2} &= \sum_{n=1}^{\infty} \frac{1}{2^n} + \frac{1}{2} \\ &= 1 + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots \\ &> 1 + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots \end{aligned}$$

We can repeat the process for the second Mersenne prime $PM_2 = 7$. Here, $\mathrm{Y} \cdot \frac{1}{f(PM_1)} = 1 + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots$ is multiplied by $\frac{1}{PM_2+1} = \frac{1}{8}$, resulting in:

$$\frac{1}{8} \cdot \frac{1}{f(PM_1)} \cdot \mathrm{Y} = \frac{1}{8} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \dots.$$

Another subtraction yields:

$$\begin{aligned} \frac{1}{f(PM_1)} \cdot \mathrm{Y} - \frac{1}{8} \cdot \frac{1}{f(PM_1)} \cdot \mathrm{Y} &= \left(1 - \frac{1}{8}\right) \cdot \frac{1}{f(PM_1)} \cdot \mathrm{Y} \\ &= \frac{1}{f(PM_2)} \cdot \frac{1}{f(PM_1)} \cdot \mathrm{Y} \\ &= 1 + \frac{1}{4} + \frac{1}{16}. \end{aligned}$$

Following the same logic, we can deduce that:

$$
\begin{aligned}
1+\frac{1}{4}+\frac{1}{16} &= \sum_{n=1}^{\infty}\frac{1}{2^n}+\frac{1}{4}+\frac{1}{16} \\
&= 1+\frac{1}{4}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}+\dots \\
&> 1+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}+\dots
\end{aligned}
$$

where $PM_3 = 31$. This process can be iterated for each Mersenne prime assuming there are infinitely many. We are able to apply this iteration because of there are no two consecutive Mersenne numbers M_n and M_{n+1} which can simultaneously be both primes for $n > 2$: Remember that if a Mersenne number M_n is prime, then n is also prime. We arrive at:

$$
\Upsilon \cdot \prod_{n=1}^{\infty}\frac{1}{f(PM_n)} = \Upsilon \cdot \frac{1}{\prod_{n=1}^{\infty} f(PM_n)} > 1
$$

which is the same as

$$
\frac{1}{\prod_{n=1}^{\infty} f(PM_n)} > \frac{1}{2}.
$$

On the one hand, suppose that the infinite product $\prod_{n=1}^{\infty} f(PM_n)$ converges. By the Euclid-Euler theorem, we consider the sequence TM_n of powers of two such that $f(TM_k \cdot PM_k) = 2$ and PM_k is the kth Mersenne prime. Thus, we have

$$
\begin{aligned}
\frac{1}{\prod_{n=1}^{\infty} f(PM_n)} &= \frac{f(TM_1)}{f(TM_1)\cdot\prod_{n=1}^{\infty} f(PM_n)} \\
&= \frac{f(TM_1)}{f(TM_1)\cdot f(PM_1)\cdot\prod_{n=2}^{\infty} f(PM_n)} \\
&= \frac{f(TM_1)}{f(TM_1\cdot PM_1)\cdot\prod_{n=2}^{\infty} f(PM_n)} \\
&= \frac{f(TM_1)}{2\cdot\prod_{n=2}^{\infty} f(PM_n)} \\
&= \frac{f(TM_1)}{2}\cdot\frac{1}{\prod_{n=2}^{\infty} f(PM_n)} \\
&= \frac{f(TM_1)}{2}\cdot\frac{f(TM_2)}{f(TM_2)\cdot\prod_{n=2}^{\infty} f(PM_n)} \\
&= \frac{f(TM_1)}{2}\cdot\frac{f(TM_2)}{f(TM_2)\cdot f(PM_2)\cdot\prod_{n=3}^{\infty} f(PM_n)} \\
&= \frac{f(TM_1)}{2}\cdot\frac{f(TM_2)}{f(TM_2\cdot PM_2)\cdot\prod_{n=3}^{\infty} f(PM_n)} \\
&= \frac{f(TM_1)}{2}\cdot\frac{f(TM_2)}{2\cdot\prod_{n=3}^{\infty} f(PM_n)} \\
&= \left(\prod_{n=1}^{2}\frac{f(TM_n)}{2}\right)\cdot\frac{1}{\prod_{n=3}^{\infty} f(PM_n)} \\
&= \dots \\
&= \prod_{n=1}^{\infty}\frac{f(TM_n)}{2}.
\end{aligned}
$$

In addition, we know that $f(TM_n) < 2$ for every natural number n. For that reason, we can state that

$$\begin{aligned}\prod_{n=1}^{\infty} \frac{f(TM_n)}{2} &= \frac{f(TM_1)}{2} \cdot \prod_{n=2}^{\infty} \frac{f(TM_n)}{2} \\ &< \frac{f(TM_1)}{2} \cdot \prod_{n=1}^{\infty} \frac{f(TM_{n+1})}{f(TM_n)} \\ &= \frac{1}{2} \cdot \prod_{n=1}^{\infty} \frac{f(TM_n)}{f(TM_n)} \\ &= \frac{1}{2}.\end{aligned}$$

Since that implies the inequality $\frac{1}{2} > \frac{1}{2}$ by transitivity, we reach a contradiction. Indeed, the infinite product $\prod_{n=1}^{\infty} \frac{f(TM_{n+1})}{f(TM_n)}$ converges when $\prod_{n=1}^{\infty} \frac{f(TM_n)}{2}$ converges. By Propositions 1 and 2, we notice that

$$\begin{aligned}\prod_{n=1}^{\infty} \frac{f(TM_{n+1})}{f(TM_n)} &= \left(\prod_{n=1}^{\infty} \frac{f(TM_{n+1})}{2}\right) \cdot \left(\prod_{n=1}^{\infty} \frac{2 \cdot TM_n}{2 \cdot TM_n - 1}\right) \\ &< \left(\prod_{n=1}^{\infty} \frac{f(TM_{n+1})}{2}\right) \cdot \left(\prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1}\right) \\ &= \left(\prod_{n=1}^{\infty} \frac{f(TM_{n+1})}{2}\right) \cdot \frac{\pi^2}{6} \\ &= \frac{2}{f(TM_1)} \cdot \frac{f(TM_1)}{2} \cdot \left(\prod_{n=1}^{\infty} \frac{f(TM_{n+1})}{2}\right) \cdot \frac{\pi^2}{6} \\ &< \left(\prod_{n=1}^{\infty} \frac{f(TM_n)}{2}\right) \cdot \frac{\pi^2}{3}.\end{aligned}$$

We can sustain the previous inequalities, because for every natural number $n > 1$ there always exists at least one square prime power q^2 between 2^n and 2^{n+2} by a refinement of the Bertrand's postulate. Moreover, we know that the fraction $\frac{x}{x-1}$ decreases as the real number x increases whenever $x > 1$. On the other hand, suppose that the infinite product $\prod_{n=1}^{\infty} f(PM_n)$ diverges. Under this new assumption, we know that

$$\frac{1}{\prod_{n=1}^{\infty} f(PM_n)} = 0.$$

Since that implies the inequality $0 > \frac{1}{2}$ by transitivity, we reach another contradiction. Consequently, by reductio ad absurdum, we can conclude that there are only a finite number of Mersenne primes. □

4. Discussion

The allure of Mersenne primes extends beyond perfect numbers. They hold immense value in cryptography, the science of secure communication. Additionally, searching for Mersenne primes pushes the boundaries of computational power, as checking if a vast number like $2^n - 1$ is prime requires immense processing capabilities. As we delve deeper into the world of Mersenne primes, we uncover a captivating interplay between mathematics, history, and technology. Their discovery not only sheds light on the intricate nature of prime numbers but also has practical applications in the modern world. This quest to settle the question of the infinitude of Mersenne primes is not merely an intellectual exercise. It delves into the very foundation of number theory, pushing the boundaries of our understanding of integers and their properties.

5. Conclusion

The question of Mersenne primes' infinitude remains a captivating problem in number theory. Continued research in these areas might lead to a more comprehensive understanding of perfect numbers and their connection to prime numbers. While the infinitude of perfect numbers might seem like a purely theoretical pursuit, the underlying concepts have practical applications. Perfect numbers play a role in areas like cryptography, where understanding the distribution of prime factors is crucial for secure communication. In conclusion, this proof sheds light on the limitations of these numbers while highlighting the ongoing quest to understand better whether the odd perfect numbers exist or not. The results open doors for further investigation, potentially leading to new discoveries in the fascinating realm of number theory.

References

1. Bateman, P.; Selfridge, J.; Wagstaff Jr, S.S. The Editor's Corner: The New Mersenne Conjecture. *The American Mathematical Monthly* **1989**, *96*, 125–128. https://doi.org/10.2307/2323195.
2. Hertlein, A. Robin's Inequality for New Families of Integers. *Integers* **2018**, *18*.
3. Ayoub, R. Euler and the Zeta Function. *The American Mathematical Monthly* **1974**, *81*, 1067–1086. https://doi.org/10.2307/2319041.
4. Abramowitz, M.; Stegun, I.A. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*; Vol. 55, US Government printing office, 1968.

Short Biography of Authors

Frank Vega is essentially a Back-End Programmer and Mathematical Hobbyist who graduated in Computer Science in 2007. In May 2022, The Ramanujan Journal accepted his mathematical article about the Riemann hypothesis. The article "Robin's criterion on divisibility" makes several significant contributions to the field of number theory. It provides a proof of the Robin inequality for a large class of integers, and it suggests new directions for research in the area of analytic number theory.