On Determining Higher Coefficient of a Second Order Hyperbolic Equation by the Variational Method

Hamlet Guliyev and Vusala Nasibzadeh

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Guliyev Hamlet
Doctor of Physical and Mathematical Sciences, Professor
Baku State University
Baku, Azerbaijan
hkuliyev@ rambler.ru

Nasibzadeh Vusala
Senior Teacher of Mathematical Analysis and Differential Equations
Sumgayit State University,
Sumgayit, Azerbaijan
vusala.nasibzada@sdu.edu.az

Abstract-The paper deals with an inverse problem of determining a higher coefficient of a second order hyperbolic equation. This problem is reduced to an optimal control problem and the new problem is studied by the methods of optimal control theory. Is proved existence theorem for optimal control and obtained necessary condition of optimality in the form integral inequality.

Keywords: *inverse problem, higher coefficient, optimal control, existence theorem, necessary condition for optimality.*

I. INTRODUCTION

In direct problems of theory of partial differential equations or in mathematical physics problems the functions that describe various physical phenomena as propagation of heat, sound, various vibrations, electromagnetic waves, etc. are sought. This time, the features of the medium under consideration or coefficients of equations are assumed to be known. However, just the features of medium in great majority of cases are unknown. Then there arise inverse problems in which on the information on the solution of the direct problem it is required to determine the coefficients of equations. As is known, these problems in many cases are ill-posed. But, at the same time, the desired coefficients of the equations characterize the medium under consideration. Therefore, solving inverse problems is very important both from a practical and theoretical point of view [1, 2, 3, 8, 9].

II. PROBLEM STATEMENT

Let $\Omega -$ be a bounded domain in the space R^n with a smooth boundary

Γ, $T > 0 -$ be a given number, $Q = \{(x,t): x \in \Omega, t \in (0,T)\} -$ be a cylinder in R^{n+1},

$S = \{(x,t): x \in \Gamma, t \in (0,T)\} -$ be a lateral surface of the cylinder Q.

It is reguired to determine a pair of functions $(u(x,t), \upsilon(x))$ from the conditions

$$\frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\upsilon(x)\frac{\partial u}{\partial x_i}\right) + a_0(x)u = f(x,t), (x,t) \in Q, \quad (1)$$

$$u(x,0) = u_0(x), \frac{\partial u(x,0)}{\partial t} = u_1(x), x \in \Omega \quad u|_S = 0, \quad (2)$$

$$\int_0^T K(x,t)u(x,t)dt = \varphi(x), \quad (3)$$

$\upsilon = \upsilon(x) \in V$, where

$$V = \left\{\upsilon(x) \in W_2^1(\Omega): \nu_0 \le \upsilon(x) \le \mu_0, \left|\frac{\partial \upsilon}{\partial x_i}\right| \le \mu_i, i = 1,\dots,n \, a.e. \, on \Omega\right\} - \quad (4)$$

is a given set, $\nu_0, \mu_0, \mu_1, \dots, \mu_n -$ are the given positive numbers, $a_0(x) \ge 0$,

$a_0 \in L_\infty(\Omega), f \in L_2(Q), u_0 \in \overset{0}{W}{}_2^1(\Omega), u_1 \in L_2(\Omega), K \in L_\infty(Q), \varphi \in L_2(\Omega) -$ are the given functions.

For the given function $\upsilon(x)$ the problem (1), (2) is a direct problem in the domain Q, for the unknow function $\upsilon(x)$ the problem (1)-(4) is said to be an inverse problem to the problem (1), (2). Note that for each fixed function $\upsilon(x) \in V$ the solution of the boundary value problem (1), (2) understood as a generalized solution from the space $W_{2,0}^1(Q)$ [4].

Under the solution from $W_{2,0}^1(Q)$ of the boundary value problem (1), (2) for the given function

$\upsilon \in V$ we will understand the function $u = u(x,t)$, equal to $u_0(x)$ for $t = 0$ and satisfying the integral identity

$$\int_Q \left[-\frac{\partial u}{\partial t}\frac{\partial \eta}{\partial t} + \sum_{i=1}^{n} \upsilon(x)\frac{\partial u}{\partial x_i}\frac{\partial \eta}{\partial x_i} + a_0(x)u\eta \right] dxdt - \int_\Omega u_1(x)\eta(x,0)dx = \int_Q f\eta dxdt$$

(5)

for all $\eta = \eta(x,t)$ from $W_{2,0}^1(Q)$, equal to zero for $t = T$.

From the results of [4, p.209-215] follows that under the above assumptions the boundary value problem (1), (2) for each fixed function $\upsilon \in V$ has a unique generalized solution from $W_{2,0}^1(Q)$ and the estimation

$$\|u\|_{W_2^1(Q)} \le c\left[\|u_0\|_{W_2^1(\Omega)} + \|u_1\|_{L_2(\Omega)} + \|f\|_{L_2(Q)}\right] \quad (6)$$

is valid. Here and in the sequel, by c we will denote various constants independent of the estimated quantities and admissible controls.

To the problem (1)-(4) we associate the following optimal control problem: it is reguired to minimize the functional

$$J(\upsilon) = \frac{1}{2}\int_\Omega \left[\int_0^T K(x,t)u(x,t;\upsilon)dt - \varphi(x) \right]^2 dx \quad (7)$$

under the conditions (1), (2), (4), where $u = u(x,t) = u(x,t;\upsilon) -$ is the solution of the boundary value problem (1), (2) corresponding to the function $\upsilon = \upsilon(x) \in V$.

We call the function $\upsilon(x)$ a control, the class $V -$ a set of admissible controls. There is close connection between the problems (1)-(4) and (1), (2), (4), (7) if in the problem (1), (2), (4), (7) $\min_{\upsilon \in V} J(\upsilon) = 0$, then additional integral condition (3) is fulfilled.

In futher, in order to avoid possible degeneration in the obtained we consider the following functionol condition for optimality:

$$J_\alpha(\upsilon) = J(\upsilon) + \frac{\alpha}{2}\int_\Omega \left[\upsilon^2(x) + \sum_{i=1}^{n} \left(\frac{\partial \upsilon}{\partial x_i}\right)^2 \right] dx = J(\upsilon) + \frac{\alpha}{2}\|\upsilon\|_{W_2^1(\Omega)}^2, \quad (8)$$

where $\alpha > 0$ –is a given number.

III. ON THE EXISTENCE OF THE SOLUTION TO PROBLEM (1), (2), (4), (8).

Theorem 1. Let the conditions accepted in the statement of problem (1)-(4) be fulfilled. Then the set of optimal controls in the problem (1), (2), (4), (8) $V_* = \{\upsilon_* \in V : J(\upsilon_*) = J_* = \inf_{\upsilon \in V} J(\upsilon)\}$ is non-empty, weakly compact in $W_2^1(\Omega)$ and any minimizing sequence $\{\upsilon^{(m)}\}$ weakly in $W_2^1(\Omega)$ converges to the set V_*.

Differentiability of the functional (8) and necessary condition for optimality.

Let $\psi = \psi(x,t;\upsilon)$ –be a generalized solution from $W_{2,0}^1(Q)$ of the adjoint problem

$$\frac{\partial^2 \psi}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\upsilon(x)\frac{\partial \psi}{\partial x_i}\right) + a_0\psi = -K(x,t) \times$$
$$\times \left[\int_0^T K(x,\tau)u(x,\tau;\upsilon)\,d\tau - \phi(x) \right], (x,t) \in Q, \quad (9)$$

$$\psi(x,T) = 0, \frac{\partial \psi(x,T)}{\partial t} = 0, x \in \Omega, \psi|S = 0. \quad (10)$$

Under the generalized solution of the boundary value problem (9), (10) for each fixed control $\upsilon \in V$ we will understand the function $\psi = \psi(x,t;\upsilon)$ from $W_{2,0}^1(Q)$, that equal to zero for $t = T$ and satisfying the integral identity

$$\int_Q \left[-\frac{\partial \psi}{\partial t}\frac{\partial g}{\partial t} + \sum_{i=1}^{n} \upsilon(x)\frac{\partial \psi}{\partial x_i}\frac{\partial g}{\partial x_i} + a_0\psi g \right] dxdt =$$
$$= -\int_Q K(x,t)\left[\int_0^T K(x,\tau)u(x,\tau;\upsilon)d\tau - \phi(x) \right] g(x,t)\,dxdt \quad (11)$$

for all $g = g(x,t)$ from $W_{2,0}^1(Q)$, that equal to zero for $t = 0$.

From the results of the [4, p.209-215] it follows that the boundary value problem (9), (10) for each fixed control$\upsilon(x) \in V$ has a unique generalized solution from $W_{2,0}^1(Q)$ and the estimation

$$\|\psi\|_{W_2^1(Q)} \le c\left[\|u\|_{L_2(Q)} + \|\phi\|_{L_2(\Omega)}\right].$$

is valid. Taking into account estimation (6), hence we have

$$\|\psi\|_{W_2^1(Q)} \le c\left[\|u_0\|_{W_2^1(\Omega)} + \|u_1\|_{L_2(\Omega)} + \|f\|_{L_2(Q)} + \|\phi\|_{L_2(\Omega)}\right]. \quad (12)$$

Let the generalized solutions $u = u(x,t;\upsilon)$ and $\psi = \psi(x,t;\upsilon)$ from $W_2^1(Q)$ of problem (1), (2) and (9), (10), respectively, have the derivatives

$\frac{\partial^2 u}{\partial x_i^2}, \frac{\partial^2 \psi}{\partial x_i^2}, i = 1, \dots, n,$ that belong to the space $L_2(Q)$.

(13)

Theorem 2. Let the conditions of theorem 1 and condition (13) be fulfilled. Then the functional (8) is continuously Frechet differentiable on V and its differential at the point $\upsilon \in V$ for the increment $\delta\upsilon \in W_\infty^1(Q)$ is determined by the expression

$$\langle J'_\alpha(\upsilon), \delta\upsilon \rangle = \int_\Omega \left[\int_0^T \sum_{i=1}^n \frac{\partial u}{\partial x_i} \frac{\partial \psi}{\partial x_i} dt \right] \delta\upsilon(x)\, dx + \alpha \int_\Omega \left[\upsilon \delta\upsilon + \sum_{i=1}^n \frac{\partial \upsilon}{\partial x_i} \frac{\partial \delta\upsilon}{\partial x_i} \right] dx. \qquad (14)$$

Theorem 3. Let the conditions of theorem 2 be fulfilled. Then for the optimality of the control $\upsilon_* = \upsilon_*(x) \in V$ in the problem (1), (2), (4), (8) it is necessary for the inequality

$$\int_\Omega \left[\int_0^T \sum_{i=1}^n \frac{\partial u_*}{\partial x_i} \frac{\partial \psi_*}{\partial x_i} dt \right] \left(\upsilon(x) - \upsilon_*(x) \right) dx + \alpha \int_\Omega \left[\upsilon_*(x) \left(\upsilon(x) - \upsilon_*(x) \right) + \sum_{i=1}^n \frac{\partial \upsilon_*}{\partial x_i} \left(\frac{\partial \upsilon(x)}{\partial x_i} - \frac{\partial \upsilon_*(x)}{\partial x_i} \right) \right] dx \geq 0 \qquad (15)$$

to hold for any $\upsilon = \upsilon(x) \in V$, where $u_* = u(x, t; \upsilon_*)$ and $\psi_* = \psi(x, t; \upsilon_*)$ – are the solutions of the problem (1), (2) and (9), (10) respectively for $\upsilon = \upsilon_*(x)$.

IV. CONCLUSION

In this work considered one inverse problem at defining of higher coefficient for hyperbolic equation. The problem is reduced to the optimal control problem and new problem is studied with methods of optimal control theory.

Is proved existence theorem for optimal control and obtained necessary condition of optimality in the form integral inequality.

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