Topological bisection method for solving of system of two equation

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Abstract.

In this article they are describe a generalization for a bisection method for solving a system of two equations using a winding number calculation.

Key words: bisection, system of equations, roots, winding number, search, topological

Introduction.

Find a method for the stable root search of a system of two equations.

For one equation, the most stable method is bisection.
For a two-equation system, all existing methods are very sensitive to the selection of an iteration starting point.
That is why we developed a new method of solving of such equations, which is the generalization of the bisection method for a single dimension.
This method is based on using a winding number of a curve that is around the point for the detection of a root inside this curve.

Topological Approach to Root Search.

Let us consider the following system of two equations,

$$f(x,y) = 0 \quad (1)$$

$$g(x,y) = 0$$

where f and g are continuous functions. Let us also consider some closed curve in the plane $x0y$.

This method is based on the mapping of a plane $x0y$ to another plane $u0v$ by the transformation T, which is defined by the formulas:

$$u = f(x,y) \quad (2)$$

$$v = g(x,y)$$

Select any circle C on plane $x0y$, calculate its mapping $K = T(C)$to plane $u0v$ and use curve K to find if a root is present inside the circle C. (1)
For this we calculate the winding number W of curve K relative to the point (0,0). If $W \neq 0$, the root of the system (1) is inside the circle. The winding number is defined in [1], section 17.
It is based on the theorem 18.1 from [1]. For a better understanding of the theorem, we re-define the notation as:

> Let T be a mapping of disk D onto the plane $u0v$. Let Cbe a boundary circle of D. Take a point (x,y) on the plane x0y and define its mapping (u,v) = T(x,y). If the point (u,v) is not on the boundary of curve T(C) and the winding number of curve T(C) about point (u,v) is not 0 when the point (x,y) is inside disk D.

In this method, we select point (u,v) to equal (0,0). If the winding number of curve T(C) relative to the point (0,0) is not equal to 0, we have a root of the system (1) inside disk D.

For a topologist, the root is the place where a vector field is equal to 0 (look at [2], chapter 2).

Topological Bisection Algorithm Description.

Let us consider the following transformation:

$$u = f(x, y)$$

$$v = g(x, y)$$

Select a circle C of radius R around the point (x0, y0) in the plane x0y.
Note the result of the circle transformation as curve K is in u0v. Calculate the winding number of the curve K relative to point (0, 0). If this number is not 0, it means that there is a root inside circle C, else we will increase R until the root will be inside of the circle.
After that, search for another smaller circle D with radius r with the same center which is without a root. This mean that a root is in the ring between R and D.

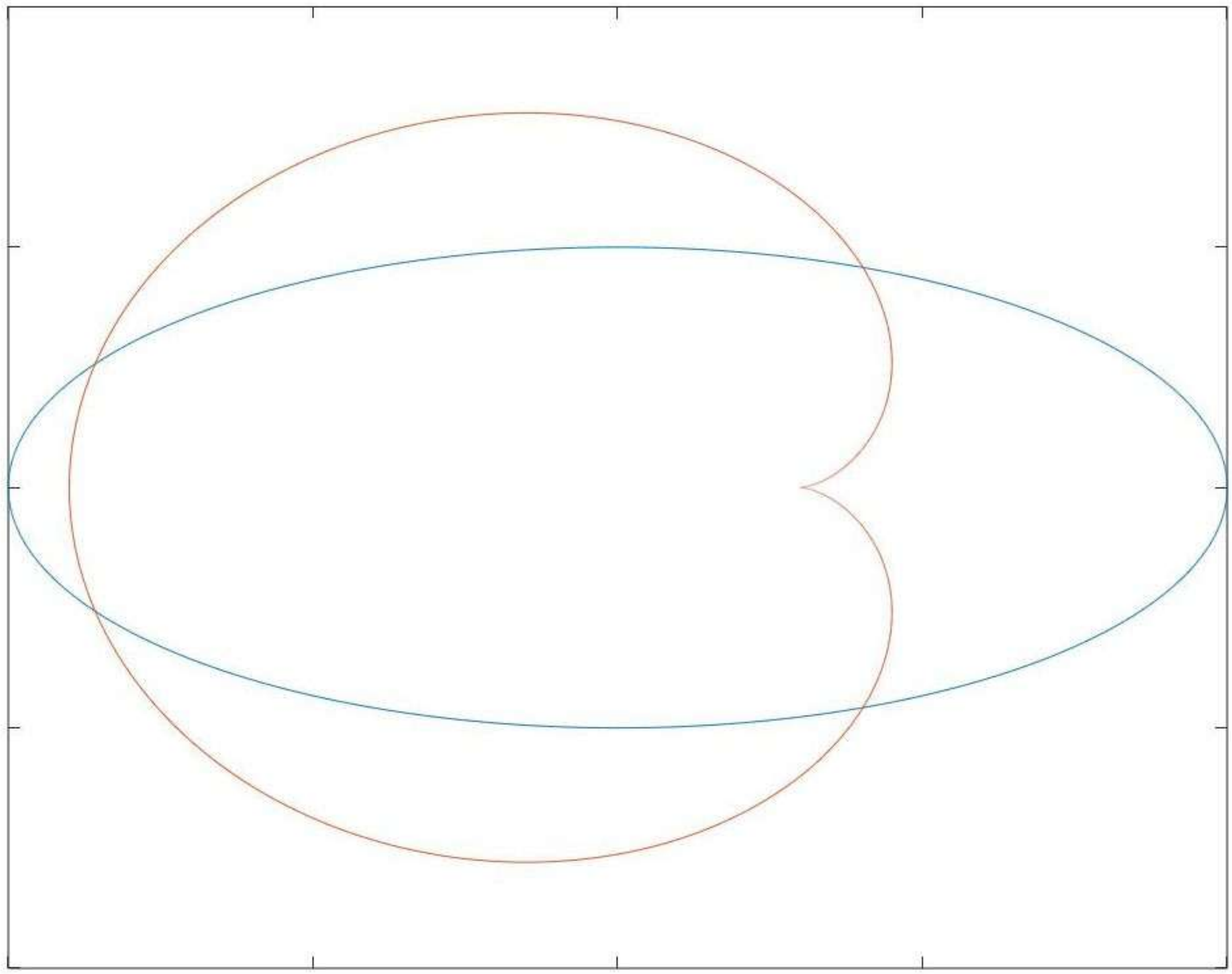
Now we can begin the bisection.
Take testing circle T with radius that is equal to (R+r) / 2.
If inside T there is a root, new C = T, else new D = T. In any step, the root is between “left” circle D and “right” circle C.
Division is repeated until circles C and D are close but never overlap.
After that, the search is finished and the searching point (x, y) on circles C and D is the coordinate closest to the root. For this, the value (f(x,y)^2+g(x,y)^) is being used which should be the minimum of the distance between the root and circle.
The corresponding point is accepted as a root estimation.

Example of using the Topological Bisection Algorithm

Let us consider the following system,

$$f(x,y) = b^2 * x^2 + a^2 * y^2 - a^2$$
(5)
$$g(x,y) = (x^2 + y^2)^2 + 4cx(x^2 + y^2) - 4c^2y^2$$

where the first formula represents an ellipse and second formula represents cardioid.
Here is a graphing of them:



The roots are the points where these curves intersect.
Calculation were made for a^2=10, b^2=5, c^2=3
We begin from selection of circle with a point in the center at (5,5) and the radius of the circle around it to be radius r = 5. We select point (5, 5) because it becomes clear that root is inside this circle.
Here are results for corresponding circles with center (5,5):

R=5	W=1
R=2.5	W=0
R=3.75	W=0
R=4.4	W=0

R=4.7 W=1
R=4.55 W=0
R=4.625 W=0
R=4.66 W=1
R=4.64 W=0

For the last two circles, the difference between their radii is small. After studying all points on these circles, the point (1.44, 2.00) is found to have been the best root estimation.

Future Directions.

It is possible to find the root faster using a bisection non-concentric circle chain.

Approach, which is described here, can also be used for solving systems of three equations.

Conclusion.

This method is the best for root search in cases where:
- there is no info about the position of the root
- a search happens in areas where functions f, g are very sensitive to x, y changes

Therefore, it is very important in the root search of equations for technical applications.

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