VIRAS: Conflict-Driven Quantifier Elimination for Integer-Real Arithmetic (Extended Version)

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Abstract

We introduce Virtual Integer-Real Arithmetic Substitution (Viras), a quantifier elimination procedure for deciding quantified linear mixed integer-real arithmetic problems. Viras combines the framework of virtual substitutions with conflict-driven proof search and linear integer arithmetic reasoning based on Cooper's method. We demonstrate that Viras gives an exponential speedup over state-of-the-art methods in quantified arithmetic reasoning, proving problems that SMT-based techniques fail to solve.

This paper is the extended version of paper "VIRAS: Conflict-Driven Quantifier Elimination for Integer-Real Arithmetic" published at LPAR 2024, written by the same authors. This version provides proofs and additional examples in order to illustrate various formal results.

1 Introduction

Automated reasoning is routinely used in applications of mathematical theory formalisation [9], formal verification [10] and web security [6]. The demand for proving properties with both quantifiers and theories is increasing in these and similar domains, especially in the context of arithmetic reasoning. Common approaches addressing this demand implement incomplete heuristics for quantifier instantiation (QI) [3, 18, 12] or integrate complete quantifier elimination (QE) [4, 2], adjusted for a particular arithmetic domain. In this paper, we improve the state-of-the-art in quantifier elimination by introducing a new calculus for *mixed* integer-real arithmetic, while aiming at reducing computational cost of QE [20].

QE transforms first-order formulas $\exists x.\phi$ or $\forall x.\phi$ into an equivalent formula ϕ' that does not contain the variable x. Seminal works solving QE were introduced within Cylindrical Algebraic Decomposition – CAD [5, 1], lazy model enumaration [17] and virtual substitution [11, 19] for non-linear real arithmetic and Cooper's method for linear integer arithmetic [7]. These techniques have been used and extended with tailored solutions for satisfiability modulo theory (SMT) solving in non-linear and linear real arithmetic (Nra, Lra) [13, 4] or linear integer arithmetic (Lia) [14, 2]. Yet, existing solutions [20, 18] fail deciding the mixed theory of linear integer and real arithmetic (Lira) adequately. The work of [20] requires formula normalizations that result in an exponential blow-up in the input formula size, whereas [18] is restricted to $\forall\exists$ problems.

This paper describes the Viras method for solving linear integer-real arithmetic formulas with *arbitrary quantifier alternations* (Sect. 4), using virtual substitutions to implement quantifier elimination in Lira. Within Viras, we combine real and integer arithmetic via a floor function $\lfloor\cdot\rfloor$ for rounding reals to closest integers. Viras uses *virtual substitutions* to eliminate quantified variables x by instantiating with so-called *virtual terms*. We extend the framework of virtual substitutions with so-called $\mathbb{Z}$-terms, allowing us to generalize Cooper's method from LIA to LIRA, and further optimizing it for equality literals (Sect. 5). VIRAS overcomes the burden of arithmetic normalisations performed in [20] and avoids an exponential blow-up in processing LIRA formula (Sect. 5). We further extend VIRAS with conflict-driven proof search (Sect. 6), by generalizing [15] to handle virtual terms involving infinitesimals ε and $\pm\infty$.

Our contributions. In summary, this paper brings the following contributions.

- We present the VIRAS method implementing a quantifier elimination procedure for linear mixed integer-real arithmetic, generalizing both Cooper's method [7] and virtual substitutions [19] by introducing $\mathbb{Z}$-terms (Sect. 4), and prove that VIRAS is indeed a quantifier elimination procedure in Theorem 1 [1].
- We show VIRAS is exponentially faster than related techniques [20]. Moreover, VIRAS can solve problems that SMT-based solutions fail to solve (Sect. 5).
- We enhance VIRAS with conflict-driven proof search, by extending the framework introduced in [15] to support ε and ∞-terms (Sect. 6).

2 Motivating Example

We illustrate LIRA reasoning and the main steps of VIRAS using the formula:

$$\exists x.\phi \quad = \quad \exists x.(\underbrace{\lfloor a \rfloor + \tfrac{1}{3} \leq x}_{L_1} \wedge \underbrace{x \leq \lfloor a \rfloor + \tfrac{2}{3}}_{L_2} \wedge \underbrace{\lceil x \rceil - x \geq c}_{L_3}) \tag{1}$$

where $\lfloor a \rfloor$ denotes the floor of the real number a; that is, the greatest integer such that $\lfloor a \rfloor \leq a$. Eliminating the quantifier $\exists x$ in the LIRA formula (1) comes with the challenge or reasoning floor-expressions within real-integer linear arithmetic.

Note that the literals L_1, L_2 impose respectively lower and upper bounds on the quantified variable x. Intuitively, L_1, L_2 imply that, in order for ϕ to hold for some x, x must be within the non-empty interval $\left[\lfloor a \rfloor + \frac{1}{3}, \lfloor a \rfloor + \frac{2}{3}\right]$. Further, literal L_3 asserts that x is in a periodically repeating set of solutions, for the following reason: as $\lceil x \rceil - x$ can be only within $[0, 1)$, the literal L_3 cannot hold if c belongs to the interval $[1, \infty)$; if $c \in [-\infty, 1)$ then L_3 holds iff $x \in \bigcup_{z \in \mathbb{Z}} (z, z+1-c]$. As such, the LIRA formula (1) holds iff intersection I of the intervals restricting the values of x, as asserted by L_1, L_2, L_3, is non-empty. Following upon this observation, I is clearly non-empty when $c < 0$. On the other hand, if $c \in [0, 1)$, then I is non-empty iff $(\lfloor a \rfloor, \lfloor a \rfloor \ + 1 - c] \cap \left[\lfloor a \rfloor + \frac{1}{3}, \lfloor a \rfloor + \frac{2}{3}\right]$ is non-empty, which is the case iff $\lfloor a \rfloor + \frac{1}{3} \leq \lfloor a \rfloor + 1 - c$ In summary, this means a quantifier-free equivalent formula to the LIRA formula (1) is $c \leq \frac{2}{3}$.

Note that by finding a quantifier-free formula $c \leq \frac{2}{3}$ equivalent to formula (1), we applied QE to (1) using arithmetic reasoning with floor-expressions. For automating such a QE process, our VIRAS method implements the following steps. We transform (1) into an equivalent, quantifier-free formula by computing a so-called *elimination set* $\mathsf{elim}(\phi)$ and by *virtually substituting* x with each element of $\mathsf{elim}(\phi)$, allowing us to replace the existentially quantified formula (1) with the following finite disjunction:

$$\exists x.\phi \quad \Longleftrightarrow \quad \phi[\![x /\!/ \lfloor a \rfloor + \tfrac{1}{3}]\!] \vee \phi[\![x /\!/ -\infty]\!] \vee \phi[\![x /\!/ \mathbb{Z}]\!] \vee \phi[\![x /\!/ \mathbb{Z} + \varepsilon]\!] \tag{2}$$

[1] Proofs are given in the appendix.

where $\phi[\![x \mathbin{/\!\!/} t]\!]$ denotes the formula obtained from ϕ by virtually substituting x with t. Note that the elements of $\mathsf{elim}(\phi)$ used for substituting x are not just regular terms, but so-called *virtual terms* that include additional symbols: ε for infinitesimal quantities, ∞ for infinity and $\mathbb{Z}$ for periodically repeating solutions. While ε and ∞ are also used [16], the periodic solutions $\mathbb{Z}$-terms are both unique and crucial for VIRAS. We immediately eliminate the symbols $\varepsilon, \infty, \mathbb{Z}$ in virtual substitutions, so that result of $\phi[\![x \mathbin{/\!\!/} t]\!]$ is a formula using the original signature of ϕ. Concretely, $t + \varepsilon$ is eliminated by replacing $\phi[\![x \mathbin{/\!\!/} t + \varepsilon]\!]$ by its limit value $\lim_{x \to t^+} \phi$; ∞ is eliminated via substituting with a constant greater than any other term, and $t + p\mathbb{Z}$ is eliminated by choosing a sufficient subset $\mathsf{fin}^{\phi}_{t+p\mathbb{Z}}$ of $\{t + pz \mid z \in \mathbb{Z}\}$ for substituting. In the case of this example a sufficient subset $\mathsf{fin}^{\phi}_{\mathbb{Z}}$ of $\mathbb{Z}$ is $\{\lfloor a \rfloor + 1\}$, the integer closest to the lower bound $L_1 = \lfloor a \rfloor + \frac{1}{3} \leq x$. As such, we apply virtual substitution:

$$\begin{aligned}
\phi[\![x \mathbin{/\!\!/} \lfloor a \rfloor + \tfrac{1}{3}]\!] &= \underbrace{\lfloor a \rfloor + \tfrac{1}{3} \leq \lfloor a \rfloor + \tfrac{1}{3}}_{\top} \wedge \underbrace{\lfloor a \rfloor + \tfrac{1}{3} \leq \lfloor a \rfloor + \tfrac{2}{3}}_{\top} \wedge \underbrace{\lceil \lfloor a \rfloor + \tfrac{1}{3} \rceil - \lfloor a \rfloor - \tfrac{1}{3} \geq c}_{\frac{2}{3} \geq c} \\
\phi[\![x \mathbin{/\!\!/} -\infty]\!] &= \bot \wedge (L_2 \wedge L_3)[\![x \mathbin{/\!\!/} -\infty]\!] \\
\phi[\![x \mathbin{/\!\!/} \mathbb{Z}]\!] &= \textstyle\bigvee_{t \in \mathsf{fin}^{\phi}_{\mathbb{Z}}} \phi[\![x \mathbin{/\!\!/} t]\!] = \phi[\![x \mathbin{/\!\!/} \lfloor a \rfloor + 1]\!] = \underbrace{\lfloor a \rfloor + 1 \leq \lfloor a \rfloor + \tfrac{2}{3}}_{\bot} \wedge (L_1 \wedge L_3)[\![x \mathbin{/\!\!/} \lfloor a \rfloor + 1]\!] \\
\phi[\![x \mathbin{/\!\!/} \mathbb{Z} + \varepsilon]\!] &= \textstyle\bigvee_{t \in \mathsf{fin}^{\phi}_{\mathbb{Z}+\varepsilon}} \phi[\![x \mathbin{/\!\!/} t]\!] = \phi[\![x \mathbin{/\!\!/} \lfloor a \rfloor + 1 + \varepsilon]\!] \\
&= \lfloor a \rfloor + 1 + \varepsilon \leq \lfloor a \rfloor + \tfrac{2}{3} \wedge (L_1 \wedge L_3)[\![x \mathbin{/\!\!/} \lfloor a \rfloor + 1 + \varepsilon]\!] \\
&= \underbrace{\lfloor a \rfloor + 1 < \lfloor a \rfloor + \tfrac{2}{3}}_{\bot} \wedge (L_1 \wedge L_3)[\![x \mathbin{/\!\!/} \lfloor a \rfloor + 1 + \varepsilon]\!]
\end{aligned}$$

allowing us to reduce (2) to $c \leq \frac{2}{3}$ as the quantifier-free equivalent of the LIRA formula (1).

3 Preliminaries

We assume familiarity with multi-sorted first-order logic and respectively denote rationals, integers and reals by $\mathbb{Q}, \mathbb{Z}$ and $\mathbb{R}$. We consider the mixed first-order theory of linear integer and real arithmetic (LIRA), corresponding to first-order logic with predicate symbols $<, \leq, \geq, >, \approx$; function symbols $+$, $q\cdot$ for $q \in \mathbb{Q}$ and $\lfloor \cdot \rfloor$; and a constant symbol 1, interpreted over $\mathbb{R}$. The function symbols $q\cdot$ are called *numerals*. A term $q \cdot (t)$ is *interpreted* as the term t multiplied by q. For simplicity, we omit parenthesis and $\cdot$ whenever it is clear from context; for example, write $3t$ for $3 \cdot (t)$. We write k for $k \cdot (1)$, $+t$ for $1t$ and $-t$ for $-1t$. By $\approx$ we denote the *equality predicate*. We write $l \not\approx r$ for $\neg(l \approx r)$. The *floor function* $\lfloor \cdot \rfloor$ applied to a term t returns the greatest integer less than or equal to t; hence, $\lfloor t \rfloor \leq t$. The *ceiling function* can be defined as $\lceil x \rceil = - \lfloor -x \rfloor$. While LIRA theory does not contain a dedicated sort of integers, it handles integer properties via the $\lfloor \cdot \rfloor$ function. Linear real arithmetic (LRA) is an instance of LIRA, without the floor function $\lfloor \cdot \rfloor$, interpreted over the reals $\mathbb{R}$. Linear integer arithmetic (LIA), also known as Presburger arithmetic, restricts LRA to the integer numerals of $\mathbb{Z}$ and is interpreted over $\mathbb{Z}$ instead of $\mathbb{R}$.

Let $\mathbf{V}$ and $\mathbf{T}$ denote respectively the set of LIRA variables and terms. We write a, b, c, x, y, z for variables; s, t, u for terms; j, k, q, p for numerals; L for literals; ϕ, ψ for formulas, all possibly with indices. We denote by $\pm$ a symbol in $\{+, -\}$ and write $\mp$ for the respective other symbol; $\diamond$ for a predicate in $\{\approx, \not\approx, >, \geq\}$; and $\gtrsim$ for a predicate in $\{>, \geq\}$. Note that all variables that are not explicitly quantified are considered parameters (i.e., implicitly universally quantified). We write $s \trianglelefteq t$ for s being a subterm of t and $s \triangleleft t$ for s being a strict subterm of t. An expression E is a term, literal or formula. We write $E[x/t]$ for the result of substituting x

by t in E; whenever it is clear from context, we write $E[t]$ for $E[x/t]$. For a formula ϕ we write $\forall\phi$ and $\exists\phi$ for the universal and existential closure of ϕ. For a set E, we write $\bigwedge E$ for $\bigwedge_{e\in E} e$; similarly for $\bigvee, \bigcap, \bigcup$ and Σ. For a formula ϕ and a first-order interpretation $\mathcal{I}$, the set $\mathsf{solSet}^{\phi}_{x,\mathcal{I}} = \{x \in \mathbb{R} \mid \mathcal{I} \models \phi[x]\}$ is the *solution set* of ϕ with respect to $\mathcal{I}$ and x. If ϕ is a conjunction of literals we write $L \in \phi$ to denote that L is a literal of ϕ.

We consider rational numbers $\frac{j}{k} \in \mathbb{Q}$ to be *normalized* such that the greatest common divisor of j, k satisfies $\gcd(j,k) = 1$. Let $\mathsf{den}(\frac{j}{k}) = k$ denote the *denominator* and $\mathsf{num}(\frac{j}{k}) = j$ the *numerator* of $\frac{j}{k}$. By $\mathsf{sgn}(q)$ we denote the *sign* of the number q with $\mathsf{sgn}(q) \in \{0, -, +\}$. We respectively introduce a generalized *quotient* function and *remainder* function as $\mathsf{quot}_p(t) = \lfloor \frac{t}{p} \rfloor$ and $\mathsf{rem}_p(t) = t - p \cdot \mathsf{quot}_p(t)$, both defined over $\mathbb{R}$. *Divisibility constraints* are expressed in LIRA as $q \mid t \iff \mathsf{rem}_q(t) \approx 0$, whereas congruence classes are defined as $s \equiv_q t \iff \mathsf{rem}_q(s) \approx \mathsf{rem}_q(t)$. We generalize least common multipliers lcm to be used with arbitrary finite sets of rationals $Q \subset \mathbb{Q}$ with $0 \notin Q$, as follows: $\mathsf{lcm}^{\mathbb{Q}}(Q) = \frac{lcm\{\mathsf{num}(q) \mid q\in Q\}}{gcd\{\mathsf{den}(q) \mid q\in Q\}}$. Clearly, for all $q \in Q$, we have $\frac{\mathsf{lcm}^{\mathbb{Q}}(Q)}{q} \in \mathbb{Z}$. We use $\llparenthesis$ and $\rrparenthesis$ as variables for interval bounds: $\llparenthesis$ is either [or (; and $\rrparenthesis$ is] or). For example the interval $\llparenthesis l, r]$ could either be $(l, r]$ or $[l, r]$, depending on $\llparenthesis$.

4 VIRAS: Virtual Integer Real Arithmetic Substitution

We now introduce the VIRAS method that performs quantifier elimination (QE) on LIRA formulas, by implementing virtual substitutions over integer-real arithmetic. Given a quantified formula $\exists x.\phi$, VIRAS translates $\exists x.\phi$ into an equivalent quantifier-free formula ϕ', which in the case of a formula where all variables in ϕ are bound means ϕ' is ground and thus can be simply be evaluated (and solved). As we can perform quantifier elimination recursively, universal quantifiers can be expressed in terms of existential ones, and existential quantifiers can be distributed over disjunctions, in the sequel we consider arbitrarily fixed $\exists x.\phi$ formula, where ϕ is a conjunction of literals, that may contain free variables that are considered parameters (i.e., implicitly universally quantified).

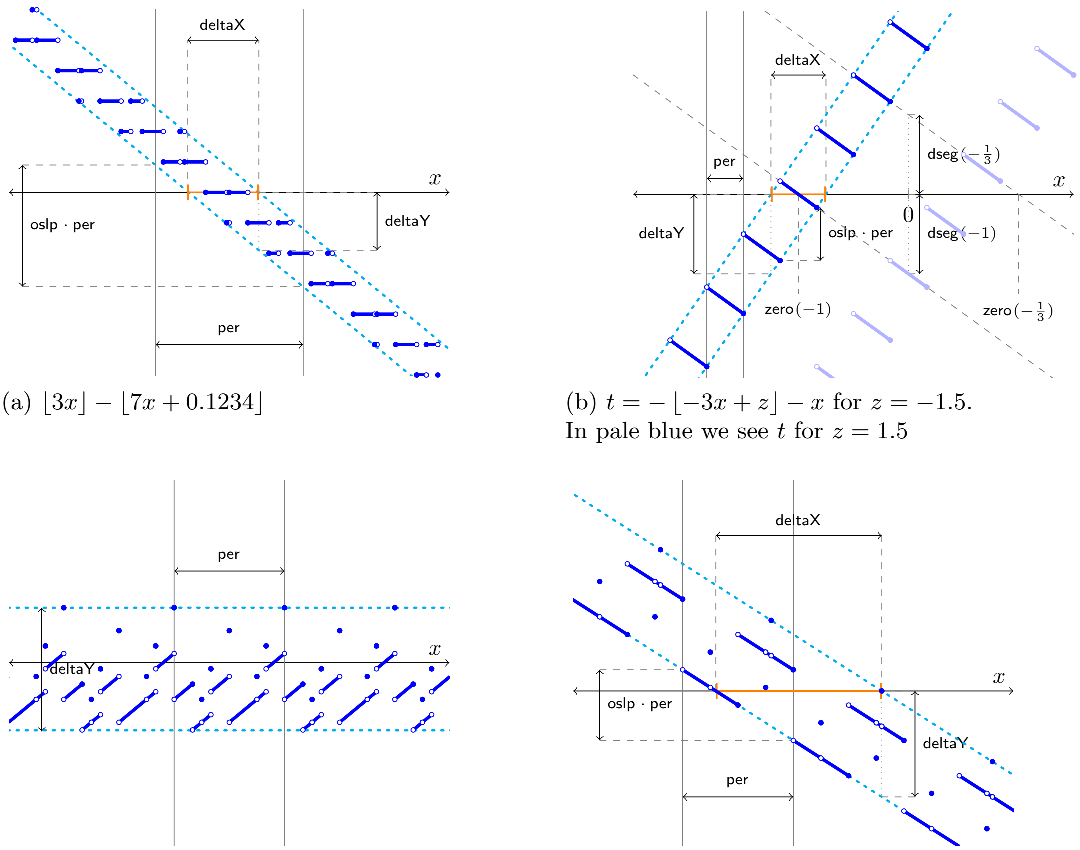
Following the setting of virtual substitutions [11, 19], VIRAS computes a finite but sufficient number of witnesses for $\exists x$ and turns the quantified formula $\exists x.\phi$ into an equaivalent finite disjunction

$$\bigvee_{t\in\mathsf{elim}^x(\phi)} \phi[\![x /\!/ t]\!],$$

where $\phi[\![x /\!/ t]\!]$ is obtained from ϕ by *virtually substituting* x with the *virtual term* t that does not contain x, and $\mathsf{elim}^x(\phi)$ is the *elimination set* of ϕ.

The core idea for finiding finite sets of witnesses $\mathsf{elim}^x(\phi)$ is that every LIRA-literal $L \in \phi$ defines a set of solution intervals. Thus if L holds for some x, then x must be contained in some solution interval S of L, thus L must also hold for the lower bound of S. As ϕ is a conjunction of such literals $\exists x.\phi[x]$ holds iff ϕ holds for any of the lower bounds of its literals. Thus we can choose the set of all lower bounds of all solution intervals as elimination set $\mathsf{elim}^x(\phi)$.

For finding the lower bounds of these solution intervals we introduce key properties of LIRA terms and literals in Sect. 4.1. As seen in our motivating example in Sect. 2 solution intervals are not only left-closed (e.g., $[l, r]$) but may also be left-open (e.g., $(l, r]$, $(-\infty, r]$), and may be periodically repeating (e.g., $\cup_{z\in\mathbb{Z}}[l + 2z, r + 2z]$). Thus we do not only substitute with regular but with virtual terms, to include lower bounds like $l + \varepsilon$, $-\infty$ and $l + 2\mathbb{Z}$. We formally define virtual terms and virtual substitutions in Sect. 4.2. Finally in Sect. 4.3 we combine the



(a) $\lfloor 3x \rfloor - \lfloor 7x + 0.1234 \rfloor$

(b) $t = -\lfloor -3x + z \rfloor - x$ for $z = -1.5$.
In pale blue we see t for $z = 1.5$

(c) $x + 2.7 - \lfloor x \rfloor + \frac{1}{2}\lfloor x + \frac{1}{3} \rfloor + \lfloor -\frac{1}{2}x - \frac{1}{3} \rfloor$
$+\frac{1}{2}(\mathsf{dots}_{\frac{1}{2}}(x) + \mathsf{dots}_{\frac{1}{3}}(x)) + 2 \cdot \mathsf{dots}_0(x)$

(d) $-x + 0.8 + \lfloor -x \rfloor - \lfloor -x + \frac{1}{2} \rfloor$
$+\frac{1}{2}(\mathsf{dots}_{\frac{3}{4}}(x) - \mathsf{dots}_{\frac{1}{4}}(x) + \mathsf{dots}_{\frac{1}{5}}(x))$

Figure 1: LIRA terms interpreted as functions in x, with $\mathsf{dots}_t(x) = \lfloor x + t \rfloor + \lfloor -x - t \rfloor$. The function graph is drawn thick and in blue, the function's linear bounds are given by the cyan dashed line, the core interval (Def. 7) is visualized in orange and marked with deltaX.

results about LIRA-terms and literals with the virtual substitution operation, allowing us to constructively define $\mathsf{elim}^x(\phi)$ and prove that VIRAS is a QE procedure for LIRA in Theorem 1.

4.1 LIRA Properties

Let us recall our motivating example from Sect. 2. We argued that the literal $L_3 = \lfloor x \rfloor - x \geq c$ of formula (1) has a periodic solution set of solutions $\bigcup_{z \in \mathbb{Z}} (z, z + 1 - c]$. The main idea of building our elimination sets is to cover all lower bounds of the intervals the solution set is composed of (i.e., $z + \varepsilon$ for every $z \in \mathbb{Z}$ in our example). Any LIRA-literal can be normalized to the form $t \diamond 0$ ($\diamond \in \{>, \geq, \approx, \not\approx\}$), thus we can characterize the lower bounds of solution sets by finding the zero crossings of LIRA-terms t. For finiding these we introduce relevant properties of LIRA terms and literals.

LIRA Terms. We first illustrate few LIRA terms in Fig. 1, where terms are interpreted as functions in x. Note that each function in Fig. 1 is non-linear and not continuous. Nevertheless, the LIRA terms of Fig. 1 have a linear upper and lower bound with the same slope (Lem. 2), which we call the *outer slope* oslp of terms (Def. 1). The lower and the upper bound distances from the origin are distY^- and distY^+, and their difference is $\mathsf{deltaY} \in \mathbb{Q}^{\geq 0}$ (Def. 2). Even though there is an infinite number of discontinuities in Fig. 1, the function graphs witness periodic repetition, parallely shifted along upper and lower bounds (Lem. 1). The *period* per of a LIRA term refers to the size of the repeating interval of its respective function graph. Fig. 1 also shows that, between each two discontinuities, the function is composed of linear segments (Lem. 4) with the same slope; we refer to these as *segment slopes* sslp (Def. 1). The line segment above any x-value can be described as a linear function (visualized via the thin gray dashed lines in Fig. 1.b) passing through each segment that starts at the term's limit lim^x_t (Def. 3) and is shifted by the distance $\mathsf{dseg}(x)$ (Def. 4) from the origin; thus, the line describing the segment above some value x_0 is given by $\mathsf{sslp} \cdot x + \mathsf{dseg}(x_0)$. It is easy to see that the truth value of a literal $t \diamond 0$ can only change at a discontinuity b or at the zero of some segment $\mathsf{zero}_t(b)$ (Def. 4). Therefore, only these values can be lower bounds of solution intervals of LIRA-literals, defining thus our elimination set (Fig. 2).

From Fig. 1 we can easily see that in the cases with $\mathsf{oslp} \neq 0$ (subfigures (a), (b), (d)), inequalities $t \diamond 0$ will always be constant true or false for x values outside of the linear bounding functions, while the area inside the bounding function contains a finite number of line segments, thus a finite number of intervals where $t \diamond 0$ could be true, thus a finite number of points we need to add to our elimination set. Further from Fig. 1.b we can see that if $\mathsf{oslp} = 0$, the truth value of inequalities $t \diamond 0$ will repeat at a period per. Within one period again there is only a finite number of line segments, hence a finite number of points for our elimination sets. Due to the periodic repeating nature of these solutions we will be able to define all of them in a finite set $\mathbb{Z}$-terms (Def. 8). Both of these insights will later be formalized in Lem. 6, and Lem. 5, which are necessary for defining virtual substitution and obtaining our main result Theorem 1. In order to do this we formally define all these notions below.

Most of the following definitions formalizing these observations will use term and variable subscripts, or superscripts (e.g., per^x_t). We will omit these for the term symbol t and the variable symbol x.

Definition 1 (Slope and Period)**.** *Let t be a* LIRA *term. By recursion on t, we define the* ***period*** per^x_t*,* ***outer slope*** oslp^x_t*, and* ***segment slope*** sslp^x_t *of t as:*

$$\mathsf{oslp}^x_y = \begin{cases} 1 & \textit{if } x = y \\ 0 & \textit{otherwise} \end{cases} \qquad \begin{aligned} \mathsf{oslp}^x_1 &= 0 \\ \mathsf{oslp}^x_{kt} &= k \cdot \mathsf{oslp}^x_t \end{aligned} \qquad \begin{aligned} \mathsf{oslp}^x_{s+t} &= \mathsf{oslp}^x_s + \mathsf{oslp}^x_t \\ \mathsf{oslp}^x_{\lfloor t \rfloor} &= \mathsf{oslp}^x_t \end{aligned}$$

$$\mathsf{sslp}^x_y = \begin{cases} 1 & \textit{if } x = y \\ 0 & \textit{otherwise} \end{cases} \qquad \begin{aligned} \mathsf{sslp}^x_1 &= 0 \\ \mathsf{sslp}^x_{kt} &= k \cdot \mathsf{sslp}^x_t \end{aligned} \qquad \begin{aligned} \mathsf{sslp}^x_{s+t} &= \mathsf{sslp}^x_s + \mathsf{sslp}^x_t \\ \mathsf{sslp}^x_{\lfloor t \rfloor} &= 0 \end{aligned}$$

$$\mathsf{per}^x_y = \mathsf{per}^x_1 = 0 \qquad\qquad \mathsf{per}^x_{kt} = \mathsf{per}^x_t$$

$$\mathsf{per}^x_{s+t} = \begin{cases} \mathsf{per}^x_s & \textit{if } \mathsf{per}^x_t = 0 \\ \mathsf{per}^x_t & \textit{if } \mathsf{per}^x_s = 0 \\ \mathsf{lcm}^{\mathbb{Q}}\{\mathsf{per}^x_s, \mathsf{per}^x_t\} & \textit{otherwise} \end{cases} \qquad \mathsf{per}^x_{\lfloor t \rfloor} = \begin{cases} 0 & \textit{if } \mathsf{per}^x_t = 0 = \mathsf{oslp}^x_t \\ \frac{1}{|\mathsf{oslp}^x_t|} & \textit{if } \mathsf{per}^x_t = 0 \neq \mathsf{oslp}^x_t \\ \mathsf{num}(\mathsf{per}^x_t) \cdot \mathsf{den}(\mathsf{oslp}^x_t) & \textit{otherwise} \end{cases}$$

Lemma 1 (Periodic Shift)**.** *If* $\mathsf{per}_t \neq 0$ *then* $\mathbb{R} \models \forall x, y. \big(t[x + \mathsf{per}\lfloor y \rfloor] \approx t[x] + \mathsf{oslp} \cdot \mathsf{per}\lfloor y \rfloor\big)$

Proof. See Appendix D. □

Example 1. *Consider the term* $t = -\lfloor -3x + z \rfloor - x$ *of Fig. 1.b. We have* $\mathsf{oslp} = 2$, $\mathsf{sslp} = -1$ *and* $\mathsf{per} = \frac{1}{3}$. *By increasing the value of* x *by* $\mathsf{per}\lfloor y \rfloor$, *the value of* $t[x]$ *increases by* $\mathsf{oslp} \cdot \mathsf{per}\lfloor y \rfloor$, *that is:*

$$\begin{aligned} t[x + \underbrace{\mathsf{per}}_{\frac{1}{3}} \lfloor y \rfloor] &\approx -\left\lfloor -3(x + \tfrac{1}{3}\lfloor y \rfloor) + z \right\rfloor - x - \tfrac{1}{3}\lfloor y \rfloor \\ &\approx \quad -\lfloor -3x + z \rfloor + \lfloor y \rfloor - x - \tfrac{1}{3}\lfloor y \rfloor \approx t[x] + \underbrace{\mathsf{per} \cdot \mathsf{oslp}}_{\frac{2}{3}} \lfloor y \rfloor \end{aligned}$$

Definition 2 (Bound Distance)**.** *Let* t *be a* LIRA*-term. We define* $\mathsf{distY}^{\pm}_{x,t} \in \mathbf{T}$ *and* $\mathsf{deltaY}_{x,t} \in \mathbb{Q}$ *by recursion on* t*:*

$$\begin{aligned} \mathsf{deltaY}_{x,y} &= 0 \\ \mathsf{deltaY}_{x,1} &= 0 \\ \mathsf{deltaY}_{x,kt} &= |k|\mathsf{deltaY}_{x,t} \\ \mathsf{deltaY}_{x,s+t} &= \mathsf{deltaY}_{x,s} + \mathsf{deltaY}_{x,t} \\ \mathsf{deltaY}_{x,\lfloor t \rfloor} &= \mathsf{deltaY}_{x,t} + 1 \\ \\ \mathsf{distY}^{+}_{x,t} &= \mathsf{distY}^{-}_{x,t} + \mathsf{deltaY}_{x,t} \end{aligned} \qquad \begin{aligned} \mathsf{distY}^{-}_{x,y} &= \begin{cases} 0 & \textit{if } x = y \\ y & \textit{otherwise} \end{cases} \\ \mathsf{distY}^{-}_{x,1} &= 1 \\ \mathsf{distY}^{-}_{x,kt} &= \begin{cases} k \cdot \mathsf{distY}^{-}_{x,t} & \textit{if } k \geq 0 \\ k \cdot \mathsf{distY}^{+}_{x,t} & \textit{if } k < 0 \end{cases} \\ \mathsf{distY}^{-}_{x,s+t} &= \mathsf{distY}^{-}_{x,s} + \mathsf{distY}^{-}_{x,t} \\ \mathsf{distY}^{-}_{x,\lfloor t \rfloor} &= \mathsf{distY}^{-}_{x,t} - 1 \end{aligned}$$

While the bounds $\mathsf{distY}^{\pm}$ are over-approximations of actual bounds[2], they yield linear bounds with same outer slopes. These bounds are being used to define the core interval (Def. 7), the interval in which a literal's truth value is not trivially true or false (Lem. 6). Overapproximating this interval thus only results in unnecessary instantiations of the formula ϕ which slows down proof search but does not affect soundness.

Lemma 2 (Linear Bounds)**.** $\mathbb{R} \models \forall x. \Big(\mathsf{oslp} \cdot x + \mathsf{distY}^{-} \leq t \leq \mathsf{oslp} \cdot x + \mathsf{distY}^{+} \Big)$.

Proof. See Appendix D. □

Example 2. *Recall term* $t = -\lfloor -3x + z \rfloor - x$ *from Ex. 1, with* $\mathsf{oslp} = 2$. *We have* $\mathsf{distY}^{-} = -z$ *and* $\mathsf{deltaY} = 1$, *which implies that* $2x - z \leq -\lfloor -3x + z \rfloor - x \leq 2x - z + 1$.

We next express that a function defined by a LIRA term is composed from the linear segments between two discontinuities. Therefore, we compute the upper limit lim^{x}_{t} of a LIRA term t (Def. 3) and derive each segment's distance to origin (Def. 4).

Definition 3 (Limit)**.** *The* ***limit term*** lim^{x}_{t} *of a* LIRA*-term* t *wrt* x *is defined by recursion on* t*, as:*

$$\begin{aligned} \mathsf{lim}^{x}_{y} &= y \\ \mathsf{lim}^{x}_{1} &= 1 \\ \mathsf{lim}^{x}_{kt} &= k \cdot \mathsf{lim}^{x}_{t} \end{aligned} \qquad \begin{aligned} \mathsf{lim}^{x}_{s+t} &= \mathsf{lim}^{x}_{s} + \mathsf{lim}^{x}_{t} \\ \mathsf{lim}^{x}_{\lfloor t \rfloor} &= \begin{cases} \lfloor \mathsf{lim}^{x}_{t} \rfloor & \textit{if } \mathsf{sslp}^{x}_{t} \geq 0 \\ \lceil \mathsf{lim}^{x}_{t} \rceil - 1 & \textit{if } \mathsf{sslp}^{x}_{t} < 0 \end{cases} \end{aligned}$$

We write lim_{t} *for* lim^{x}_{t} *if* x *is clear in the context.*

Example 3. *The term* $t = -\lfloor -3x + z \rfloor - x$ *of Fig. 1.b has* $\mathsf{lim}_{t} = \lfloor 3x - z \rfloor + 1 - x$.

[2] Ex. 14 in Appendix A.1 shows that finding tight(er) bounds is very expensive

Definition 4 (Segment Line). *The* ***segment distance*** $\mathsf{dseg}_t(x_0)$ *of a* LIRA*-term* t *at* x_0 *is:*

$$\mathsf{dseg}^x_t(x_0) = -\mathsf{sslp}^x_t \cdot x_0 + \mathsf{lim}^x_t[x_0] \qquad \mathsf{zero}^x_t(x_0) = x_0 - \frac{\mathsf{lim}^x_t[x_0]}{\mathsf{sslp}_t}$$

The ***segment line*** *of* t *at* x_0 *is* $\mathsf{sslp}_t \cdot x + \mathsf{dseg}_t(x_0)$*, whereas* $\mathsf{zero}_t(x_0)$ *is the* ***zero of the segment*** *of* t *at* x_0*.*

Example 4. *In* t *from Fig. 1.b, we have* $\mathsf{dseg}(b) = \lfloor 3b - z \rfloor + 1$*. Hence,* $\mathsf{dseg}(-1/3) = \lfloor -z \rfloor$*,* $\mathsf{dseg}(-1) = \lfloor -z \rfloor - 2$*, and* $\mathsf{zero}(b) = \mathsf{dseg}(b)$*. The dotted lines of Fig. 1).b show* $\mathsf{sslp} \cdot x + \mathsf{dseg}(-1/3)$ *and* $\mathsf{sslp} \cdot x + \mathsf{dseg}(-1)$*, with the corresponding zeros* $\mathsf{zero}(-1/3)$ *and* $\mathsf{zero}(-1)$*. The lines' distances to the origin are* $\mathsf{dseg}(-1/3)$ *and* $\mathsf{dseg}(-1)$*.*

We next introduce the set breaks^∞ of discontinuities, which is infinite but periodically repeating. We therefore specify finite sets breaks of terms with a formal parameter $\mathbb{Z}$, capturing that if $t + p\mathbb{Z} \in \mathsf{breaks}$ then $\{t + pz \mid z \in \mathbb{Z}\} \subseteq \mathsf{breaks}^\infty$.

Example 5. *For* $-\lfloor -3x + z \rfloor - x$ *in Fig. 1.b, we have* $\mathsf{breaks}^\infty = \{\frac{z}{3} + \frac{i}{3} \mid i \in \mathbb{Z}\}$ *and* $\mathsf{breaks}_t = \{\frac{z}{3} + \frac{1}{3}\mathbb{Z}\}$*.*

To define breaks^∞, we compute the intersection of infinite sets defined by $t + p\mathbb{Z}$ with constant-sized intervals $(\!|l, l + q\,|\!)$ $(l, t \in \mathbf{T}, q, p \in \mathbb{Q}^{\geq 0})$, using *grid instersections.*

Definition 5 (Grid Intersection). *For* $s, t \in \mathbf{T}$ *and* $p, k \in \mathbb{Q}^{>0}$*, the* ***grid intersection*** *is*

$$(s + p\mathbb{Z}) \sqcap (\!|t, t + k|\!) = \{start_{(\!|} + np \mid n \in \mathbb{N}, np <_{|\!)} k\}$$

where

$$\lceil t \rceil^{s+p\mathbb{Z}} = t + \mathsf{rem}_p(s - t) \qquad \lceil t + \varepsilon \rceil^{s+p\mathbb{Z}} = \lfloor t + p \rfloor^{s+p\mathbb{Z}} \qquad start_{[} = \lceil t \rceil^{s+p\mathbb{Z}} \qquad <_{]} \;=\; \leq$$

$$\lfloor t \rfloor^{s+p\mathbb{Z}} = t - \mathsf{rem}_p(t - s) \qquad \lfloor t - \varepsilon \rfloor^{s+p\mathbb{Z}} = \lceil t - p \rceil^{s+p\mathbb{Z}} \qquad start_{(} = \lceil t + \varepsilon \rceil^{s+p\mathbb{Z}} \qquad <_{)} \;=\; <$$

Intuitively, a term $t + p\mathbb{Z}$ is a grid that starts at value t and repeats with period p. The operation $\sqcap$ intersects this grid with an interval, whereas the operations $\lfloor s \rfloor^{t+p\mathbb{Z}}$ and $\lceil s \rceil^{t+p\mathbb{Z}}$ are rounding the grid value next to s. We thus have the following result.

Lemma 3 (Grid Intersection). $(s + p\mathbb{Z}) \sqcap (\!|t, t + k|\!) \supseteq (\{s + pz \mid z \in \mathbb{Z}\} \cap (\!|t, t + k|\!))$

Proof. See Appendix D. □

Example 6. *Consider the interval* $[a, a + 4)$ *and the grid* $1 + 2\mathbb{Z}$*. As*

$$I = (1 + 2\mathbb{Z}) \sqcap [a, a + 4) = \{\lceil a \rceil^{1+2\mathbb{Z}} + i \mid i \in \{0, 2\}\} = \{a + \mathsf{rem}_2(1 - a) + i \mid i \in \{0, 2\}\},$$

we obtain $I = \{1 - 2\left\lfloor \frac{1-a}{2} \right\rfloor, 3 - 2\left\lfloor \frac{1-a}{2} \right\rfloor\}$*. Hence, the values in* I *are in* $G = \{1 + 2z \mid z \in \mathbb{Z}\}$*. Further, since* $\mathsf{rem}_2(1 - a) \in [0, 2)$ *yields that* $\lceil a \rceil^{1+2\mathbb{Z}} = a + \mathsf{rem}_2(1 - a)$ *is the smallest value in* $G \cap [a, a + 4)$*, which means* $I \subseteq G \cap [a, a + 4)$*.*

We have now all ingredients to define the set breaks of discontinuities, using over-approximation as for linear bounds (Lem. 2).

Definition 6. *The set of* ***discontinuities*** breaks^x_t *of a* LIRA*-term* t *wrt variable* x *is defined by recursion on* t*, as:*

$$\begin{aligned} \mathsf{breaks}^x_y &= \mathsf{breaks}^x_1 = \emptyset \\ \mathsf{breaks}^x_{kt} &= \mathsf{breaks}^x_t \\ \mathsf{breaks}^x_{s+t} &= \mathsf{breaks}^x_s \cup \mathsf{breaks}^x_t \end{aligned} \qquad \mathsf{breaks}^x_{\lfloor t \rfloor} = \begin{cases} \mathsf{breaks}^x_t & \textit{if } \mathsf{sslp}_t = 0 \\ \{\mathsf{zero}_t(0) + \mathsf{per}_{\lfloor t \rfloor}\mathbb{Z}\} & \textit{if } \mathsf{breaks}^x_t = \emptyset \;\&\; \mathsf{sslp}_t \neq 0 \\ \mathsf{breaks}^x_t \cup \mathsf{breaksInSeg}^x_t & \textit{if } \mathsf{breaks}^x_t \neq \emptyset \;\&\; \mathsf{sslp}_t \neq 0 \end{cases}$$

$$\mathsf{breaksInSeg}^x_t = \left\{ b + \mathsf{per}_{\lfloor t \rfloor}\mathbb{Z} \;\middle|\; \begin{array}{ll} b \in (\mathsf{zero}(b_0) + \frac{1}{\mathsf{sslp}_t}\mathbb{Z}) \sqcap [b_0, b_0 + p^{\min}_t) & \textit{where} \\ b_0 \in (b'_0 + p\mathbb{Z}) \sqcap [b'_0, b'_0 + \mathsf{per}_{\lfloor t \rfloor}) & \textit{where} \\ b'_0 + p\mathbb{Z} \in \mathsf{breaks}_t & \end{array} \right\}$$

$$p^{\min}_t = \min\{p \mid b + p\mathbb{Z} \in \mathsf{breaks}^x_t\}$$

$$\mathsf{breaks}^{x,\infty}_t = \{t + pz \mid z \in \mathbb{Z}, t + p\mathbb{Z} \in \mathsf{breaks}^x_t\}$$

The piecewise linearity of functions defined by Lira terms is then expressed as: between any two neighbouring breaks b^+ and b^-, the term t is described by a linear function $\mathsf{sslp}\cdot x + \mathsf{dseg}(b^-)$.

Lemma 4 (Piecewise Linearity)**.** *Let $\mathcal{I}$ be an $\mathbb{R}$-interpretation, $x \in \mathbf{V}$ and t a Lira-term such that $\mathsf{breaks} \neq \emptyset$ and $b^- \in \mathsf{breaks}^\infty$. Let $b^+ = \min\{b \mid b \in \mathsf{breaks}^\infty, \mathcal{I} \models b > b^-\}$, and $\pm \in \{+, -\}$. Then*

$$\mathcal{I} \models \forall x \in (b^-, b^+), y \in [b^-, b^+).\, (t[x] \approx \mathsf{lim}_t[x] \approx \mathsf{sslp} \cdot x + \mathsf{dseg}(y))\,.$$

Proof. See Appendix D. □

LIRA Literals. Let us now introduce some key properties of Lira literals that will allow us to specify finite elimination sets. We assume Lira literals to be normalized to $t \diamond 0$, and distinguish two kinds of Lira-literals: A Lira literal is called *periodic* if $\mathsf{oslp}_t = 0$ and *aperiodic* otherwise. Periodic literals' solution sets repeat periodically, which allows us to finitely specify the lower bounds of their solution sets using $\mathbb{Z}$-terms (formally defined in Sect. 4.2), while aperiodic literals only have a finite number of solution intervals that can be found using the bounds of their so-called core interval (Def. 7).

Fig. 1.b shows that the solutions of periodic literals repeat in a periodic manner:

Lemma 5 (Periodic Literals)**.** *If $L = t \diamond 0$ is a periodic Lira-literal ($\mathsf{oslp}_t = 0$), then*

$$\mathbb{R} \models \forall y.\big(L[x] \leftrightarrow L[x + \mathsf{per}_t \lfloor y \rfloor]\big)$$

Proof. See Appendix D. □

Example 7. *Consider the literal $L_3 = t \geq 0$, with $t = \lceil x \rceil - x - c$, $\mathsf{oslp}_t = 0$ and $\mathsf{per}_t = 1$ from the motivating example of Sect. 2. We have $t[x + \lfloor s \rfloor] \approx t[x]$ for any s. Hence, $L_3[x + \lfloor s \rfloor] \leftrightarrow L_3[x]$.*

The truth values of aperiodic literals do not repeat. Instead they have a constant limit value $\mathsf{lim}^{\pm\infty}$ and a so-called *core interval.*

Definition 7 (Core Interval)**.** *Let t be a Lira-term with $\mathsf{oslp}_t \neq 0$. The **core interval** of t is $[\mathsf{distX}^-_t, \mathsf{distX}^+_t]$, where*

$$\mathsf{distX}^-_t = -\frac{\mathsf{distY}^{\mathsf{sgn}(\mathsf{oslp}_t)}_t}{\mathsf{oslp}_t} \qquad \mathsf{deltaX}_t = \frac{\mathsf{deltaY}_t}{|\mathsf{oslp}_t|} \qquad \mathsf{distX}^+_t = \mathsf{distX}^-_t + \mathsf{deltaX}_t$$

Bounds of the core intervals are given by the zeros of the linear bounds from Lem. 2. Within a core interval, a Lira literal may be evaluated to both true and false, while outside of the interval the literal's value is equal to the constant value $\mathsf{lim}^{\pm\infty}_L \in \{\top, \bot\}$, as next given.

Lemma 6 (Limit Value)**.** *If $L = t \diamond 0$ is an aperiodic* LIRA*-literal ($\mathsf{oslp}_t \neq 0$), then the values outside of the core interval of t satisfy the following:*

$$\mathbb{R} \models \forall x < \mathsf{distX}_t^-.(L[x] \leftrightarrow \mathsf{lim}_L^{-\infty}) \qquad \mathbb{R} \models \forall x > \mathsf{distX}_t^+.(L[x] \leftrightarrow \mathsf{lim}_L^{+\infty})$$

where

$$\mathsf{lim}_{t\approx 0}^{\pm\infty} = \bot \qquad \mathsf{lim}_{t\not\approx 0}^{\pm\infty} = \top \qquad \mathsf{lim}_{t\gtrsim 0}^{\pm\infty} = \pm\mathsf{oslp} > 0$$

Proof. See Appendix D. □

Example 8. *Consider again our term $t = -\lceil -3x + z\rceil - x$ and the literal $L = t > 0$. We have $\mathsf{distX}^- = \frac{z-1}{2}$, $\mathsf{deltaX} = \frac{1}{2}$, $\mathsf{lim}_L^{+\infty} = \top$ and $\mathsf{lim}_L^{-\infty} = \bot$. Therefore, L is $\bot$ for all values less than $\frac{z-1}{2}$ and $\top$ for all values greater than $\frac{z}{2}$.*

4.2 Virtual Substitutions in VIRAS

Recall that virtual substitutions do not replace variables by regular terms, but by *virtual terms* from an extended language. Formally, we have the following.

Definition 8 (Virtual Term)**.** *A **virtual term** v is a sum $t + e\varepsilon + z\mathbb{Z} + i\infty$ with $t \in \mathbf{T}, e \in \{0,1\}, z \in \mathbb{Q}^{\geq 0}, i \in \{0,+,-\}$, where $z = 0$ or $i = 0$. We may omit summands with zero coefficients. We write $\mathbb{Z}(v) = z$, $\varepsilon(v) = e$ and $\infty(v) = i$. A virtual term is **plain** if $e = z = i = 0$ and **proper** otherwise.*

The new symbols ε, $\mathbb{Z}$, ∞ do not occur in the result of applying virtual substitution. Instead, the *virtual substitution function* (Def. 9) eliminates these auxiliary symbols, as follows. As ε represents an infinitesimal quantity, we compute $L[\![x /\!/ s+\varepsilon]\!]$ by replacing it by $\lim_{x\to s^+} L$ (cases 4–5 of Def. 9). The summand ∞ represents an infinitely large constant that is divisible by every rational number. Thus we compute $\phi[\![x /\!/ t \pm \infty]\!]$ by replacing all aperiodic literals $A \in \phi$ by $\mathsf{lim}_A^{\pm\infty}$ and replacing periodic literals $P \in \phi$ by $P[\![x /\!/ t]\!]$ (case 3 of Def. 9).

Virtual terms $t + p\mathbb{Z}$ represent infinite sets of substitutions: $\phi[\![x /\!/ t + p\mathbb{Z}]\!]$ is true iff $\exists z \in \mathbb{Z}.\phi[\![x /\!/ t + pz]\!]$; hence, we compute a finite subset $\mathsf{fin}_{t+p\mathbb{Z}}^{\phi} \subset \{t + pz \mid z \in \mathbb{Z}\}$ such that $\exists z \in \mathbb{Z}.\phi[\![x /\!/ t + pz]\!] \leftrightarrow \bigvee_{t' \in \mathsf{fin}_{t+p\mathbb{Z}}^{\phi}} \phi[\![x /\!/ t']\!]$ (case 1 of Def. 9). Such a finite subset was given in Sect. 2, where $\mathsf{fin}_{\mathbb{Z}}^{\phi} = \{\lfloor a\rfloor + 1\}$, the smallest integer satisfying the lower bound $\lfloor a\rfloor + \frac{1}{3}$.

Definition 9 (Virtual Substitution)**.** *A **virtual substitution function** $\circ[\![\circ /\!/ \circ]\!]$ maps a conjunction of* LIRA*-literals, a variable, and a virtual term to a formula. We write $\phi[\![t]\!]$ for $\phi[\![x /\!/ t]\!]$. Let ϕ be a conjunction of* LIRA*-literals, t a term, v a virtual term with $\mathbb{Z}(v) = 0$, $P = \{L \in \phi \mid L \text{ is periodic }\}$, and $A = \{L \in \phi \mid L \text{ is aperiodic }\}$. Then,*

1. $\phi[\![x /\!/ t + e\varepsilon + p\mathbb{Z}]\!] = \bigvee_{t' \in \mathsf{fin}_{t+p\mathbb{Z}}^{\phi}} \phi[\![x /\!/ t' + e\varepsilon]\!]$ *where*

   V1. if $\forall L \in A.\mathsf{lim}_L^{\pm\infty} = \top$: $\quad \mathsf{fin}_{t+p\mathbb{Z}}^{\phi} = \{s \pm \infty \mid s \in \big(t + p\mathbb{Z} \sqcap [t, t+\lambda)\big) \}$

   V2. if $\exists L \in A.L = u \approx 0$: $\quad \mathsf{fin}_{t+p\mathbb{Z}}^{\phi} = \big(t + p\mathbb{Z} \sqcap [\mathsf{distX}_{u\approx 0}^-, \mathsf{distX}_{u\approx 0}^+]\big)$

   V3. otherwise: $\quad \mathsf{fin}_{t+p\mathbb{Z}}^{\phi} = \bigcup_{L \in A, \mathsf{lim}_L^{-\infty} = \bot} \big(t + p\mathbb{Z} \sqcap [\mathsf{distX}_L^-, \mathsf{distX}_L^+ + \lambda]\big)$

   $\lambda = \mathsf{lcm}^{\mathbb{Q}}(\{p\} \cup \{\mathsf{per}_L \mid L \in P\})$

2. $(\bigwedge_{L\in\phi} L)[\![x /\!/ v]\!] = \bigwedge_{L\in\phi} (L[\![x /\!/ v]\!])$

3. $(s \diamond 0)[\![x /\!/ v \pm \infty]\!] = \begin{cases} \mathsf{lim}^{\pm\infty}_{s\diamond 0} & \textit{if } s \diamond 0 \textit{ is aperiodic } (\mathsf{oslp}_s = 0) \\ (s \diamond 0)[\![x /\!/ v]\!] & \textit{if } s \diamond 0 \textit{ is periodic } (\mathsf{oslp}_s \neq 0) \end{cases}$

4. $((\neg)s \approx 0)[\![x /\!/ t + \varepsilon]\!] = \begin{cases} (\neg)\bot & \textit{if } \mathsf{sslp}_s \neq 0 \\ (\neg)\mathsf{lim}_s[t] \approx 0 & \textit{if } \mathsf{sslp}_s = 0 \end{cases}$

5. $(s \gtrsim 0)[\![x /\!/ t + \varepsilon]\!] = \begin{cases} \mathsf{lim}_s[t] \geq 0 & \textit{if } \mathsf{sslp}_s > 0 \\ \mathsf{lim}_s[t] \gtrsim 0 & \textit{if } \mathsf{sslp}_s = 0 \\ \mathsf{lim}_s[t] > 0 & \textit{if } \mathsf{sslp}_s < 0 \end{cases}$

6. $(s \diamond 0)[\![x /\!/ t]\!] = s[x/t] \diamond 0$

Note that for finding $\mathsf{fin}^{\phi}_{t+p\mathbb{Z}}$ in general (case 1 of Def. 9), we use periodic literals (Lem. 5) and core intervals (Lem. 6), as follows. Literals L with $\mathsf{lim}^{+\infty}_L = \top$ and $\mathsf{lim}^{-\infty}_L = \bot$ can only be true from the beginning of the core interval $[\mathsf{distX}^-_L, \infty)$, thus we only need to instantiate with values in this interval for every such literal. Further in the interval $(\mathsf{distX}^+, \infty)$ the literal L will always be $\mathsf{lim}^{+\infty}_L = \top$, while the truth value of periodic literals will repeat with a period of λ. Thus if there is a solution in $(\mathsf{distX}^+, \infty)$, then there must be one in $(\mathsf{distX}^+, \mathsf{distX}^+ + \lambda]$. This means it is sufficient for $\mathsf{fin}^{\phi}_{t+p\mathbb{Z}}$ to contain all values in $[\mathsf{distX}^-_L, \mathsf{distX}^+_L + \lambda] \cap \{t + pz \mid z \in \mathbb{Z}\}$ for such L. This reasoning correponds to case (V3) of Def. 9 and illustrated in Ex. 15 in Appendix A.1. The cases (V1), (V2) of Def. 9 handle formulas where there is no such literal L. The cases (V1), (V3) of Def. 9 generalize Cooper's method for LIA [7], as discussed in Sect. 5.

4.3 Quantifier Elimination via Elimination Sets

To find sufficient finite elimination sets, we proceed as follows. If there is some x such that a formula ϕ is true, then x is an element of some solution interval I of ϕ. Therefore, ϕ is true for the lower bound of I. Hence, if we take all terms that might be lower bounds of a solution interval of any literal of ϕ, we obtain an elimination set for ϕ. It is easy to see that $\exists x.\phi$ holds if there is a t such that $\phi[\![t]\!]$ holds; thus we may compute an over-approximation of the exact set of lower bounds.

Example 9. *In Sect. 2, the solution set of L_3 is $\bigcup_{z\in\mathbb{Z}}(z, z+1-c]$. Hence, we may derive an elimination set as $\{z + \varepsilon \mid z \in \mathbb{Z}\}$, which is finitely represented as $\{\mathbb{Z} + \varepsilon\}$.*

Definition 10 (Elimination Set)**.** *The* ***elimination set*** $\mathsf{elim}^x(\phi)$ *of a conjunction of literals ϕ with respect to the variable x is defined in Fig. 2.*

Let us make the following remarks upon Def. 10. If $\mathsf{breaks} = \emptyset$, we have a simple linear function; in this case, the lower bounds of the solution intervals can be computed as in LRA. For literals $t \diamond 0$ where $\mathsf{breaks}_t \neq \emptyset$, firstly notice that every discontinuity b of t can be the lower bound of a solution interval $[b, b]$. Therefore, we add ebreak to the elimination set. For periodic literals, ebreak is breaks_t a finite representation of the full infinite set of discontinuities, while for aperiodic literals we only add discontinuities within the core interval $(\mathsf{distX}^-, \mathsf{distX}^+)$. Between any two discontinuities, t can be described as segment of a linear function (Lem. 4). Therefore, we find the lower bounds eseg of the solution intervals of these segments using the zeros of the segments $\mathsf{zero}(b)$, as well as the discontinuities b bounding the segments. For periodic literals, we only consider all periodically repeating values; whereas for aperiodic literals consider those for conjunctions of literals ϕ and ψ: $\quad \mathsf{elim}^x(\phi \wedge \psi) = \mathsf{elim}^x(\phi) \cup \mathsf{elim}^x(\psi)$

if $\mathsf{breaks} = \emptyset$

$$
\begin{aligned}
&\mathsf{elim}(t \diamond 0) = \{-\infty\} && \text{if } \mathsf{sslp} = 0 \\
&\mathsf{elim}(t \not\approx 0) = \{-\infty, \mathsf{zero}_t(0) + \varepsilon\} \\
&\mathsf{elim}(t \approx 0) = \{\mathsf{zero}_t(0)\} \\
&\mathsf{elim}(t \gtrsim 0) = \begin{cases} \{\mathsf{zero}_t(0)\} & \text{if } \mathsf{sslp} > 0 \;\&\; {\gtrsim} = {\geq} \\ \{\mathsf{zero}_t(0) + \varepsilon\} & \text{if } \mathsf{sslp} > 0 \;\&\; {\gtrsim} = {>} \\ \{-\infty\} & \text{if } \mathsf{sslp} < 0 \end{cases}
\end{aligned}
$$

if $\mathsf{breaks} \neq \emptyset$

$$
\mathsf{elim}(t \diamond 0) = \begin{cases} \mathsf{ebreak} \cup \mathsf{eseg} & \text{if } t \diamond 0 \text{ is periodic} \\ \mathsf{ebreak} \cup \mathsf{eseg} \cup \mathsf{ebound}^+ \cup \mathsf{ebound}^- & \text{if } t \diamond 0 \text{ is aperiodic} \end{cases}
$$

$$
\mathsf{ebound}^+ = \begin{cases} \{\mathsf{distX}^+, \mathsf{distX}^+ + \varepsilon\} & \text{if } \mathsf{lim}^{+\infty} = \top \\ \{\mathsf{distX}^+\} & \text{if } \mathsf{lim}^{+\infty} = \bot \end{cases}
$$

$$
\mathsf{ebound}^- = \begin{cases} \{\mathsf{distX}^-, -\infty\} & \text{if } \mathsf{lim}^{-\infty} = \top \\ \{\mathsf{distX}^-\} & \text{if } \mathsf{lim}^{-\infty} = \bot \end{cases}
$$

$$
\mathsf{ebreak} = \begin{cases} \{\, b + p\mathbb{Z} \mid b + p\mathbb{Z} \in \mathsf{breaks}\} & \text{if } t \diamond 0 \text{ is periodic} \\ \bigcup\{(b + p\mathbb{Z}) \sqcap (\mathsf{distX}^-, \mathsf{distX}^+) \mid b + p\mathbb{Z} \in \mathsf{breaks}\} & \text{if } t \diamond 0 \text{ is aperiodic} \end{cases}
$$

$$
\mathsf{eseg} = \begin{cases} \{t + \varepsilon \mid t \in \mathsf{ebreak}\} & \text{if } \mathsf{sslp} = 0 \text{ or } \mathsf{sslp} < 0 \;\&\; \diamond \in \{>, \geq\} \\ \{t + \varepsilon \mid t \in \mathsf{ebreak}\} \cup \{t \mid t \in \mathsf{ezero}\} & \text{if } \mathsf{sslp} > 0 \;\&\; \diamond \in \{\geq\} \\ \{t + \varepsilon \mid t \in \mathsf{ebreak}\} \cup \{t + \varepsilon \mid t \in \mathsf{ezero}\} & \text{if } \mathsf{sslp} > 0 \;\&\; \diamond \in \{>\} \\ \{t + \varepsilon \mid t \in \mathsf{ebreak} \cup \mathsf{ezero}\} & \text{if } \mathsf{sslp} \neq 0 \;\&\; \diamond \in \{\not\approx\} \\ \mathsf{ezero} & \text{if } \mathsf{sslp} \neq 0 \;\&\; \diamond \in \{\approx\} \end{cases}
$$

$$
\mathsf{ezero} = \begin{cases} \{\mathsf{zero}(b) + p\mathbb{Z} \mid b + p\mathbb{Z} \in \mathsf{breaks}\} & \text{if } t \diamond 0 \text{ is periodic} \\ \{\mathsf{zero}(b) \mid b + p\mathbb{Z} \in \mathsf{breaks}\} & \text{if } t \diamond 0 \text{ is aperiodic} \;\&\; \mathsf{oslp} = \mathsf{sslp} \\ \bigcup\left\{ \begin{pmatrix} \mathsf{zero}(b) + (1 - \frac{\mathsf{oslp}}{\mathsf{sslp}})p\mathbb{Z} \\ \sqcap(\mathsf{distX}^-, \mathsf{distX}^+) \end{pmatrix} \middle| \, b + p\mathbb{Z} \in \mathsf{breaks} \right\} & \text{if } t \diamond 0 \text{ is aperiodic} \;\&\; \mathsf{oslp} \neq \mathsf{sslp} \end{cases}
$$

Figure 2: Definition of the elimination set elim^x computed by VIRAS.

in the core interval. Both eseg and ebreak are limited to the core interval $(\mathsf{distX}^-, \mathsf{distX}^+)$ for aperiodic literals, thus we also need to cover the lower bounds $\mathsf{ebound}^\pm$ of solution sets outside of the core interval.

Based on our definition of elimination sets and virtual substitution, we obtain the following result, asserting that elim^x can be used to eliminate existential quantifiers.

Theorem 1 (Quantifier Elimination)**.** *Let ϕ be a non-empty conjunction of* Lira*-literals.*

$$\mathbb{R} \models \exists x.\phi \leftrightarrow \bigvee_{t \in \mathsf{elim}(\phi)} \phi[\![t]\!]$$

Proof. See Appendix D. □

Example 10. *Consider formula ϕ from Sect. 2, with*

$$\begin{aligned}\mathsf{elim}^x(\phi) &= \mathsf{elim}^x(\lfloor a \rfloor + \tfrac{1}{3} \leq x) \wedge \mathsf{elim}^x(x \leq \lfloor a \rfloor + \tfrac{2}{3}) \wedge \mathsf{elim}^x(\lfloor x \rfloor - x \geq c) \\ &= \mathsf{elim}^x(\underbrace{x - \lfloor a \rfloor - \tfrac{1}{3}}_{t_1} \geq 0) \wedge \mathsf{elim}^x(\underbrace{\lfloor a \rfloor + \tfrac{2}{3} - x}_{t_2} \geq 0) \wedge \mathsf{elim}^x(\underbrace{\lfloor x \rfloor - x - c}_{t_3} \geq 0)\end{aligned}$$

As $\mathsf{breaks}_{t_1} = \mathsf{breaks}_{t_2} = \emptyset$*, we compute the elimination sets* $\mathsf{elim}^x(t_1 \geq 0)$ *and* $\mathsf{elim}^x(t_2 \geq 0)$*, resulting in* $\mathsf{elim}^x(t_1 \geq 0) = \{-\lfloor a \rfloor - \frac{1}{3}\}$ *and* $\mathsf{elim}^x(t_2 \geq 0) = \{-\infty\}$*.*

For $\mathsf{elim}^x(t_3 \geq 0)$*, we have* $\mathsf{breaks}^x_{t_3} = \{\mathbb{Z}\}$*. As $t_3 \geq 0$ is periodic (*$\mathsf{oslp}_{t_3} = 0$*), the elimination set* $\mathsf{elim}^x(t_3 \geq 0)$ *consists of all discontinuities* $\mathsf{ebreak} = \mathsf{breaks} = \{\mathbb{Z}\}$ *and* eseg*. The intuition of* eseg *is the least value t within two breaks $t \in (b^-, b+)$ for which $t_3 \geq 0$ can hold. As the slope of the segment is negative* $\mathsf{sslp}_{t_3} = -1$*, this value must be $b^- + \varepsilon$. Therefore,* $\mathsf{eseg} = \{b + \varepsilon \mid b \in \mathsf{breaks}\} = \{\mathbb{Z} + \varepsilon\}$*. Thus, we derive* $\mathsf{elim}^x(\phi) = \{-\lfloor a \rfloor - \frac{1}{3}, -\infty, \mathbb{Z}, \mathbb{Z} + \varepsilon\}$*.*

5 VIRAS and Related Methods

We discuss and highlight the main differences of Viras compared to the state-of-the-art algorithms in solving quantified linear arithmetic problems. In a nutshell, we generalize Cooper's method [7] for Lia to be used with Lira while allowing for additional optimization for equality literals. Further, virtual substitutions in Viras yield an exponential speed-up compared to the method for solving Lira described by [20]. As a result and thanks to its Lira reasoning, Viras solves problems that state-of-the-art SMT techniques [8, 3] fail to solve.

VIRAS Generalizations upon Cooper's Method. While Cooper's method [7] implements a QE procedure only for Lia, our Viras calculus solves full Lira formulas[3]. Similarly to Viras splitting literals into periodic P and aperiodic literal A (Def. 9.1), Cooper's method splits a formula $\phi = \mathcal{L} \wedge \mathcal{U} \wedge \mathcal{D}$ into literals capturing lower bounds $\mathcal{L}$, upper bounds $\mathcal{U}$ and divisibility constraints $\mathcal{D}$. The solution to $\mathcal{D}$ are found by (i) computing λ, the *lcm* of all divisibility constraints, and (ii) instantiating $\mathcal{D}$ with one number for every congruence class modulo λ. For also solving $\mathcal{L} \wedge \mathcal{U}$, the formula ϕ is instantiated with $\{l, \ldots, l + \lambda - 1\}$ for every lower bound $x \geq l \in \mathcal{L}$. Generalizing Cooper's method to Lira is however not straightforward, as bounds and equivalence classes over $\mathbb{R}$ differ from the ones over $\mathbb{Z}$. While Cooper's method requires that a solution to $\mathcal{D}$ is one of the congruence classes $\{0 \ldots \lambda - 1\}$, as proper real numbers ($\mathbb{R} \setminus \mathbb{Z}$) cannot be captured by these equivalence classes. In Viras, we therefore compute equivalence classes of solutions using elim over $\mathbb{Z}$-terms (e.g. using $\frac{1}{2} + 2\mathbb{Z}$). We compute values of these equivalence classes to be the closest values the lower bound literals L ($L \in A, \mathsf{lim}^{-\infty} L = \bot$ in item (V3) of case 1 of Def. 9), using core intervals of these L and λ. Cooper's method contains an optimization for formulas where $\mathcal{L}$ or $\mathcal{U}$ is empty, we implement this optimization in (V1). An additional optimization offered by Viras that is not present in Cooper's method is (V2).

[3] Note that Lia (aka. Presubrger Arithmetic) is sometimes defined with an auxiliary divisibility predicate $q \mid t$ (read "q divides t). This predicate can be expressed in Lira as $\exists x.q \lfloor x \rfloor \approx t$.

Exponential Speed-Up of VIRAS. The work of [20] provides a quantifier elimination procedure for LIRA based on the following idea. A variable x is split into its integer $\lfloor x \rfloor$ and fractional $x - \lfloor x \rfloor$ parts, allowing for the separate uses of external QE procedures for LIA and LRA, respectively. Doing so, formula preprocessing comes with a heavy normalization burden: formulas are normalized such that their literals are of the form $jx + k \lfloor x \rfloor + t \diamond 0$, where $x \not\trianglelefteq t$. With such normalizations, Ex. 11 shows an exponential blow-up in the formula size. Unlike this, VIRAS does not use external QE procedures but operates directly on LIRA terms, implementing virtual substitutions.

Example 11. *We illustrate the* VIRAS *benefits in avoiding the expensive normalizations of [20]. Let $n \in \mathbb{N}^{>0}$ and $t_i \in \mathbf{T}$ such that $x \not\trianglelefteq t_i$. Consider the formula $\phi = \sum_{i=1}^{n} \lfloor 2x + t_i \rfloor \approx 0$.*

The work of [20] normalizes ϕ to $\phi' = \bigvee_{j_1=0}^{2} \dots \bigvee_{j_n=0}^{2} \sum_{i=1}^{n} (2 \lfloor x \rfloor + j_i + \lfloor t_i \rfloor) \approx 0$ and eliminates quantifiers of ϕ'. Note that the size of ϕ' is $O(3^n)$ in the size of ϕ.

In contrast, VIRAS *computes the elimination set of ϕ as $\mathsf{elim}(\phi) = (\mathsf{ebreak}) \cup \mathsf{eseg} \cup \mathsf{ebound}^+ \cup \mathsf{ebound}^-$ where $|\mathsf{ebound}^+ \cup \mathsf{ebound}^-|$ is $O(1)$ and $|(\mathsf{ebreak})|$ is $O(|\mathsf{breaks}| 2\mathsf{deltaX})$ in the size of ϕ. As $\mathsf{breaks} = \{-\frac{t_i}{2} + \mathbb{Z} \mid i \in \{1 \dots n\}\}$ and $\mathsf{deltaX} = 2n$, we derive $|(\mathsf{ebreak})|$ being $O(n^2)$ in the size of ϕ. Further, the size of eseg is $O([\mathsf{ebreak}]) = O(n^2)$. Using Theorem 1,* VIRAS *thus solves ϕ exponentially faster compared to [20].*

Solving Quantified SMT Problems in LIRA. Thanks to its sound and complete LIRA reasoning, VIRAS solves LIRA problems that existing SMT techniques fail to solve, like the example below.

Example 12. *Consider the formula:*

$$\forall x, z. \Big(\underbrace{\lceil x + z \rceil > \lfloor x + z \rfloor}_{L_1} \wedge \underbrace{\lceil z \rceil \approx \lfloor z \rfloor}_{L_2} \rightarrow \underbrace{\lfloor x \rfloor \not\approx x}_{L_3} \Big) \tag{3}$$

Literal L_1 is true iff $x + z$ is not an integer, L_2 is true iff z is an integer and L_3 is true iff x is not an integer. Formula (3) *thus captures that, if the sum $x + z$ is not an integer and z is an integer, then x cannot be an integer, which is clearly valid. Existing SMT techniques [8, 3] fail to solve* (3)*, whereas* VIRAS *can easily prove*[4] (3).

Conflict-Driven Reasoning. A complementary approach to VIRAS comes with conflict-driven proof search for arithmetic reasoning [15]. Within [15], validity of $\exists x_1 \dots x_n.\phi$ is (dis)proved, where ϕ is a conjunction of literals. Therefore the algorithm attempts at building a satisfying assignment $x_1 \leftarrow t_1 \dots x_n \leftarrow t_n$ using terms from the elimination set for t_i. If the assignment makes ϕ true, $\exists\phi$ must be valid. Whenever some partial assignment makes ϕ false, we speak of a *conflict*. Then a lemma is learned to block the generation of such a conflicting assignment, and proof search backtracks. When no more backtracking is possible, ϕ is unsatisfiable. While learning lemmas is central in [15], the approach is limited to elimination sets with plain virtual terms, that is to virtual terms not containing ε or $\pm\infty$, which is essential for VIRAS. In Sect. 6 we generalize lemma learning from [15], allowing us to handle proper virtual terms and improve VIRAS with conflict-driven proof search.

[4] see Appendix A.2

6 Conflict-Driven VIRAS

For extending VIRAS with conflict-driven lemma learning during proof search as introduced in [15], we need to resolve the following limitation. In case [15] identifies an assigment $x \leftarrow t$ as a conflict, the approach will introduce a lemma $x \not\approx t$ [5] to exclude this assignment. Simply using this approach in VIRAS is not sufficient since [15] can not handle the assigments of $x \leftarrow t + \varepsilon$, as $x \not\approx t + \varepsilon$ is not a formula in our LIRA signature. To address this limitation, we introduce a function lemma_ϕ (Def. 12) to generate lemmas that exclude assignments for arbitrary virtual terms t from Def. 8; in particular, we generate ε-lemmas and ∞-lemmas. In order for the calculus using the lemma function lemma_ϕ to be sound, we impose that, if $\neg\phi[\![x \mathbin{/\!/} t]\!]$ and $\phi[x]$, then $\mathsf{lemma}_\phi(x \not\approx t)$ (Lem. 7.1). Further, to ensure completeness of lemma_ϕ, we exclude the current assignment $\neg\mathsf{lemma}_\phi(x \not\approx t)[\![x \mathbin{/\!/} t]\!]$ (Lem. 7.2). In what follows, we formalize this setting, allowing us to integrate VIRAS with conflict-driven proof, resulting in our improved CD-VIRAS calculus for QE over LIRA formulas $\exists x.\phi$.

ε-Lemmas We first focus on finding a lemma that is false when we virtually substitute x by $t + \varepsilon$; we denote such lemmas as ε-lemmas. For these lemmas, we can use any formula $x \leq t \vee u < x$ for any $u > t$. To find such a u reason as follows. If ϕ does not hold at some point $t+\varepsilon$, there must be some non-empty interval (t, u) where ϕ does not hold. ϕ can only change its truth value at a value v when one of its literals $s \diamond 0$ changes its truth value at v. For deriving an ε-lemma, we use Fig. 3 to compute $\mathsf{nxt}^{\top}_{s\diamond 0}(t + \varepsilon)$ as an overapproximation of the set of all such v. In particular, if $\mathsf{breaks}_s = \emptyset$, the truth value of $s \diamond 0$ can only change from false to true if the linear function defined by $t[x]$ intersects with zero. Note that literals $-x > 0$ can only change their truth values from true to false, but not from false to true; hence $\mathsf{nxt}^{\top}_{-x>0}(t) = \emptyset$. If $\mathsf{breaks}_s \neq \emptyset$, then $s \diamond 0$ can only change its truth value if either the line segment of s at point $t + \varepsilon$ intersects with zero ($\mathsf{curZero}(s + \varepsilon)$ in Fig. 3) or at the next discontinuity b where that segment ends ($\mathsf{nextBreak}(s + \varepsilon)$ in Fig. 3). Based on this reasoning, we introduce formula $\mathsf{inFalseInterval}^{\phi}_{t+\varepsilon}(x)$ to define an interval I with lower bound $t + \varepsilon$, that includes only values for which ϕ is false (given $\phi[\![t + \varepsilon]\!]$ is false).

Definition 11 (False Interval)**.** *Let ϕ be a conjunction of literals. The **false interval** of ϕ at $t + \varepsilon$ is denoted as $\mathsf{inFalseInterval}^{\phi}_{t+\varepsilon}(x)$ and defined in Fig. 3.*

Example 13. *Consider the formula $\phi = L_1 \wedge L_2$, where $L_1 = \lceil x \rceil - \lfloor x \rfloor - \frac{1}{2} \geq 0$ and $L_2 = x - \lfloor x - \frac{1}{2} \rfloor + 1 > 0$, and we want to find a lemma to exclude an assignment $\langle\rangle \mid x \leftarrow \varepsilon$. In this case we have $\mathsf{nxt}^{\top}_{L_1}(\varepsilon) = \mathsf{nextBreak} = \{1\}$, and $\mathsf{nxt}^{\top}_{L_2}(\varepsilon) = \mathsf{nextBreak}(\varepsilon) \cup \mathsf{curZero}(\varepsilon) = \{\frac{1}{2} + \varepsilon\} \cup \{0\}$. This means that*

$$\begin{aligned}
\mathsf{inFalseInterval}^{\phi}_{\varepsilon}(x) =& 0 < x \wedge \bigwedge_{e \in \mathsf{nxt}^{\top}_{L_1}(\varepsilon)} (\varepsilon \lessdot e \rightarrow x \lessdot e) \wedge \bigwedge_{e \in \mathsf{nxt}^{\top}_{L_2}(\varepsilon)} (\varepsilon \lessdot e \rightarrow x \lessdot e) \\
=& 0 < x \wedge (\varepsilon \lessdot 1 \rightarrow x \lessdot 1) \wedge \big((\varepsilon \lessdot \frac{1}{2} + \varepsilon \rightarrow x \lessdot \frac{1}{2} + \varepsilon) \wedge (\varepsilon \lessdot 0 \rightarrow x \lessdot 0)\big) \\
=& 0 < x \wedge (0 < 1 \rightarrow x < 1) \wedge \big((0 < \frac{1}{2} \rightarrow x \leq \frac{1}{2}) \wedge (0 < 0 \rightarrow x < 0)\big) \\
\iff& 0 < x \wedge x < 1 \wedge x \leq \frac{1}{2} \iff 0 < x \wedge x \leq \frac{1}{2}
\end{aligned}$$

So we derive the lemma $x \leq 0 \vee \frac{1}{2} < x$.

[5] To see how derivations and lemma derival works in detail see Ex. 16 and 17 in Appendix B.2

$$\mathsf{inFalseInterval}^{\phi}_{t+\varepsilon}(x) = (t < x \wedge \bigwedge_{L\in\phi} \bigwedge_{e\in \mathsf{nxt}^{\top}_{L}(t+\varepsilon)} (t+\varepsilon \lessdot e \to x \lessdot e))$$

$$s+\varepsilon \lessdot t \quad = s < t \qquad s \lessdot t \quad = s < t$$

$$s+\varepsilon \lessdot t+\varepsilon = s < t \qquad s \lessdot t + \varepsilon = s \leq t$$

if $\mathsf{breaks} = \emptyset$

$$\mathsf{nxt}^{\top}_{t\diamond 0}(s+\varepsilon) = \emptyset \quad \text{if } \mathsf{sslp} = 0$$

$$\mathsf{nxt}^{\top}_{t\approx 0}(s+\varepsilon) = \{\mathsf{zero}_t(0)\} \qquad \mathsf{nxt}^{\top}_{t\gtrsim 0}(s+\varepsilon) = \begin{cases} \{\mathsf{zero}_t(0)\} & \text{if } \mathsf{sslp} > 0 \;\&\; \gtrsim \,=\, \geq \\ \{\mathsf{zero}_t(0)+\varepsilon\} & \text{if } \mathsf{sslp} > 0 \;\&\; \gtrsim \,=\, > \\ \emptyset & \text{if } \mathsf{sslp} < 0 \end{cases}$$

$$\mathsf{nxt}^{\top}_{t\not\approx 0}(s+\varepsilon) = \emptyset$$

if $\mathsf{breaks} \neq \emptyset$

$$\mathsf{nxt}^{\top}_{t\diamond 0}(s+\varepsilon) = \mathsf{nextBreak}(s) \quad \text{if } \mathsf{sslp} = 0$$

$$\mathsf{nxt}^{\top}_{t\diamond 0}(s+\varepsilon) = \mathsf{nextBreak}(s) \cup \mathsf{curZero}(s+\varepsilon) \quad \text{if } \mathsf{sslp} \neq 0$$

$$\mathsf{nextBreak}(s) = \{\lceil s+\varepsilon \rceil^{b+p\mathbb{Z}} \mid b+p\mathbb{Z} \in \mathsf{breaks}\}$$

$$\mathsf{curZero}(s) = \begin{cases} \{\mathsf{zero}_t(s)+\varepsilon\} & \text{if } \diamond = > \;\&\; \mathsf{sslp} > 0 \\ \{\mathsf{zero}_t(s)\} & \text{if } \diamond = \geq \;\&\; \mathsf{sslp} > 0 \text{ or } \diamond \,=\, \approx \\ \emptyset & \text{if } \diamond \in \{>, \geq\} \;\&\; \mathsf{sslp} < 0 \text{ or } \diamond \,=\, \not\approx \end{cases}$$

Figure 3: Definition of $\mathsf{inFalseInterval}^{\phi}_{t+\varepsilon}(x)$.

∞-Lemmas We next derive lemmas to exclude assignments using virtual terms containing $\pm\infty$; we refer to these lemmas as ∞-lemmas. For $\phi[\![x /\!/ t+\infty]\!]$ to be false, there are two options: (i) either one of its aperiodic literals L has a limit $\mathsf{lim}^{\pm\infty}_{L} = \bot$, or (ii) one of its periodic literal L is false at t. For (i), we simply derive the ∞-lemma of $x \leq \mathsf{distX}^{+}_{L}$ or $\mathsf{distX}^{-}_{L} \leq x$. For (ii), our ∞-lemma has to exclude the solution t. A naïve approach would derive the ∞-lemma of $x \not\approx t$; this lemma however not suffice as $\mathsf{lim}^{\pm\infty}_{x\not\approx t} = \top$, hence $(x \not\approx t)[\![t \pm \infty]\!] = \top$. Therefore, we need to find some periodic literal that excludes the solution t. As L is periodic, we have $L[\![t]\!] \leftrightarrow L[\![t + \lambda \lfloor z \rfloor]\!]$, thus we obtain the ∞-lemma $\mathsf{rem}_{\lambda}(x) \approx \mathsf{rem}_{\lambda}(t)$, which is equaivalent to $x \approx t + \lambda(\mathsf{quot}_{\lambda}(x) - \mathsf{quot}_{\lambda}(t))$; this ∞-lemma is to be used for any t that does not contain ε. With a similar reasoning for $t + \varepsilon + \infty$ and by using ε-lemmas, we derive the ∞-lemma $\neg\mathsf{inFalseInterval}^{\phi}_{t+\lambda(\mathsf{quot}_{\lambda}(x)-\mathsf{quot}_{\lambda}(t))}(x)$.

$\mathbb{Z}$-flattening Lemmas for virtual terms $t+p\mathbb{Z}$ could be computed similarly to ∞ lemmas using $\mathsf{rem}_p(t) \not\approx \mathsf{rem}_p(x)$. Nevertheless, virtual substitutions with $\mathbb{Z}$-terms pose another challenge: as [15] transforms literals into disjunctions, the assumption of ϕ being a conjunction of literals required by the conflict-driven framework is violated. We resolve this difficulty by transforming the elimination set elim^x into the flattened version $\mathsf{elim}^x_{\mathsf{flat}}(\phi) = \{t \mid t + 0\mathbb{Z} \in \mathsf{elim}^x(\phi)\} \cup \bigcup\{\mathsf{fin}^{\phi}_{t+p\mathbb{Z}} \mid t + p\mathbb{Z} \in \mathsf{elim}^x(\phi), p \not\approx 0\}$. It is easy to see that $\bigvee_{t\in\mathsf{elim}^x_{\mathsf{flat}}(\phi)} \phi[\![t]\!] = \bigvee_{t\in\mathsf{elim}^x(\phi)} \phi[\![t]\!]$, hence $\mathsf{elim}^x_{\mathsf{flat}}(\phi)$ fulfils Theorem 1 as well, but does not contain any $\mathbb{Z}$ terms. Therefore we use $\mathsf{elim}^x_{\mathsf{flat}}$ instead of elim^x, allowing us to only deal with conjuctions of literals and replacing the need to generate lemmas for $\mathbb{Z}$-terms.

Leaf Conflict

$$(F, S, L) \vdash (F, S, L \cup \{\bigvee_{i=1}^{k} \mathsf{lemma}_F(x_i \not\approx t_i)\})$$
$$\quad where S = \langle x_1 \leftarrow t_1, \ldots, x_k \leftarrow t_k \rangle$$
if $F \mathbin{/\!\!/} S$ is trivally inconsistent and $L \mathbin{/\!\!/} S$ is not trivially inconsistent

Inner Conflict

$$(F, S \mid x_k \leftarrow \bot, L) \vdash (F, S \mid x_k \leftarrow \bot, L \cup \{\bigvee_{i=1}^{k-1} \mathsf{lemma}_F(x_i \not\approx t_i)\})$$
$$\quad where S = \langle x_1 \leftarrow t_1, \ldots, x_{k-1} \leftarrow t_{k-1} \rangle$$
if $L \mathbin{/\!\!/} S$ is not trivially inconsistent

Figure 4: Rules of the CD-Viras calculus, that differ from those of CDVS.

CD-VIRAS By using ε-lemmas, ∞-lemmas and $\mathbb{Z}$-flattening, we combine Viras with conflict-driven proof search, resulting in our CD-Viras calculus. Doing so, we ajdust only two rules from [15], namely Inner Conflict and Leaf Conflict as named in [15]. Instead of lemmas $\bigvee_{i \in I} x_i \not\approx t_i$ introduced by these rules in [15], in CD-Viras we use the lemmas $\bigvee_{i \in I} \mathsf{lemma}_\phi(x_i \not\approx t_i)$ using the lemma function lemma_ϕ defined below. The modified rules are given in Fig. 4.

Definition 12 (CD-Viras Lemmas). *Let ϕ be a conjunction of literals, t a term, and $A = \{L \mid L \in \phi, \mathsf{oslp}_L \neq 0\}$. The **lemma function** of* CD-Viras *is defined as: conflicts are:*

$$
\begin{aligned}
\mathsf{lemma}_\phi\ (x \not\approx t) &= x \not\approx t \\
\mathsf{lemma}_\phi\ (x \not\approx t + \varepsilon) &= \neg \mathsf{inFalseInterval}^{\phi}_{t+\varepsilon}(x) \\
\mathsf{lemma}_\phi\ (x \not\approx t + e\varepsilon + \infty) &= x \leq \mathsf{distX}^{+}_{L} && \textit{if } \mathsf{lim}^{+\infty}_{L} = \bot \textit{ for some } L \in A \\
\mathsf{lemma}_\phi\ (x \not\approx t + e\varepsilon - \infty) &= \mathsf{distX}^{-}_{L} \leq x && \textit{if } \mathsf{lim}^{-\infty}_{L} = \bot \textit{ for some } L \in A \\
\mathsf{lemma}_\phi\ (x \not\approx t \pm \infty) &= \mathsf{rem}_\lambda(x) \not\approx \mathsf{rem}_\lambda(t) \\
\mathsf{lemma}_\phi\ (x \not\approx t + \varepsilon \pm \infty) &= \neg \mathsf{inFalseInterval}^{\phi}_{t+\lambda(\mathsf{quot}_\lambda(x) - \mathsf{quot}_\lambda(t))+\varepsilon}(x)
\end{aligned}
$$

Soundness and completeness of the lemma function is established next, yielding that the calculus CD-Viras itself is sound and complete.

Lemma 7. *Let ϕ be a conjunction of literals and v be a virtual term with $\mathbb{Z}(v) = 0$. Our function lemma_ϕ satisfies the following properties:*

1. $\neg \phi[\![x \mathbin{/\!\!/} v]\!] \to \forall x(\phi \to \mathsf{lemma}_\phi(x \not\approx v))$. *(soundness)*
2. $\neg \mathsf{lemma}_\phi[\![x \mathbin{/\!\!/} v]\!]$. *(completeness)*

Proof. See Appendix D. □

Using Lem. 7 soundness and completeness of the calculus CD-Viras can be proven in the same way as in [15]. Lem. 7.1 is needed for soundness, while Lem. 7.2 is needed for completeness. [6]

[6] For a detailed explanation of the proof we refer to Appendix B.3.

7 Conclusion

We introduce the VIRAS calculus as a new quantifier elimination procedure for solving quantifier formulas with mixed linear integer-real arithmetic. VIRAS uses virtual substitutions and can be integrated with conflict-driven proof search. Computing more accurate bounds $\mathsf{distY}^{\pm}$, as well as more accurate discontinuity sets breaks, is an interesting line for future research, with the purpose of more efficient proof search. Implementing VIRAS is another challenge for further work. We pointed out that our method gives an exponential speed-up over [20] for some classes of formulas. Nevertheless finding actual complexity bounds for our method remains for future research.

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A Additional examples

A.1 Virtual Integer Real Arithmetic Substitution

We give addition examples for Sect. 4.

LIRA-Terms

Example 14. *Let* $\mathsf{in}\mathbb{Z}(x) = 1 - \lceil x \rceil + \lfloor x \rfloor$*. This function is* 1 *for* $x \in \mathbb{Z}$ *and* 0 *otherwise. Consider* $s = \mathsf{in}\mathbb{Z}(x+t) - \mathsf{in}\mathbb{Z}(x+u)$*. There are two cases: Either* $t - \lfloor t \rfloor \approx u - \lfloor u \rfloor$*, which means that* $s = 0$ *for all* x*, or* s *can take each of the values in* $\{0, 1, -1\}$ *depending on* x*. This means depending on* t *and* u *the tightest* deltaY *can be either 0 or 2. Thus finding the tightest value for* deltaY *requires checking validity for of formula* $t - \lfloor t \rfloor \approx u - \lfloor u \rfloor$*.*

Further this example demonstrates that finding the exact set of discontinuities is hard. Depending on whether $t - \lfloor t \rfloor \approx u - \lfloor u \rfloor$, the discontinuities of s could be $\{-t + \mathbb{Z}, u + \mathbb{Z}\}$ or $\emptyset$.

Virtual Substitution We give the following example case (V3) in Def. 9.

Example 15. *Consider* $\phi = t > 0 \land x < 0 \land \mathsf{rem}_3(x) \approx \mathsf{rem}_3(c) \land \mathsf{rem}_2(x) \not\approx 1$ *with* $t = -\lfloor -3x + z \rfloor - x$ *as in our examples from before. When computing the elimination set we will get the virtual term* $c + 3\mathbb{Z}$*. We split the formula into all aperiodic literals* $t > 0 \land x < 0$*, and all periodic literals* $P = \mathsf{rem}_3(x) \approx \mathsf{rem}_3(c) \land \mathsf{rem}_2(x) \not\approx 1$*. If there is some* $t' = c + 3\mathbb{Z}$ *such that* $\phi[\![t']\!]$*, then, due to Lem. 6* t' *has to be either in the core interval* $[\frac{z-1}{2}, \frac{z}{2}]$ *of* $t > 0$*, or it has to be greater than* $\mathsf{distY}^+_t = \frac{z}{2}$ *to make* $t > 0$ *true.*

Due to Lem. 5 we know that the truth value of all literals in P *will repeat if we add or subtract* $\lambda = \mathsf{lcm}^{\mathbb{Q}}\{\mathsf{per}_L \mid L \in P\} = 6$ *to* t'*. Therefore if there is some* $t' \in (\mathsf{distX}^+_t, \infty)$*, where* ϕ *holds, then there must also be such a* $t'' \in (\mathsf{distX}^+_t, \lambda]$*.*

Combining this we know that it is sufficient to substitute ϕ *with* $\{c + 3z \mid z \in \mathbb{Z}\} \cap [\frac{z-1}{2}, \frac{z}{2} + 6]$ *which – as we know by Lem. 3 – can be over-approximated using* $(c + 3\mathbb{Z}) \sqcap [\frac{z-1}{2}, \frac{z}{2} + 6] = \{c - \lfloor \frac{c}{3} - \frac{z}{6} + \frac{1}{6} \rfloor, c - \lfloor \frac{c}{3} - \frac{z}{6} + \frac{1}{6} \rfloor + 3, c - \lfloor \frac{c}{3} - \frac{z}{6} + \frac{1}{6} \rfloor + 6\}$*.*

A.2 Solving Quantified SMT Problems

We demonstrate how VIRAS proofs validity of the formula ϕ from *Sect.* 5. Let us now see how VIRAS solves this problem. In a first step we normalize the problem to be able to apply quantifier elimination:

$$
\begin{aligned}
&\phi = \forall x, z.\Big(\lceil x + z \rceil > \lfloor x + z \rfloor \land \lceil z \rceil \approx \lfloor z \rfloor \to \lfloor x \rfloor \not\approx x \Big) \\
\iff &\neg \exists x, z. \neg \Big(\lceil x + z \rceil > \lfloor x + z \rfloor \land \lceil z \rceil \approx \lfloor z \rfloor \to \lfloor x \rfloor \not\approx x \Big) \\
\iff &\neg \exists x, z. \Big(\lceil x + z \rceil > \lfloor x + z \rfloor \land \lceil z \rceil \approx \lfloor z \rfloor \land \lfloor x \rfloor \approx x \Big) \\
\iff &\neg \exists x. \exists z. \Big(\underbrace{\lceil x + z \rceil - \lfloor x + z \rfloor}_{t_1} > 0 \land \underbrace{\lceil z \rceil - \lfloor z \rfloor}_{t_2} \approx 0 \land \underbrace{\lfloor x \rfloor - x}_{t_3} \approx 0 \Big)
\end{aligned}
$$

Then it will compute the elimination set for z to eliminate the first quantifier $\exists z$. $\mathsf{elim}^z(\phi) = \mathsf{elim}^z(t_1 > 0) \cup \mathsf{elim}^z(t_2 \approx 0) \cup \mathsf{elim}^z(t_3 \approx 0)$. We have $\mathsf{oslp}_z(t_1) = \mathsf{oslp}_z(t_2) = \mathsf{oslp}_z(t_3) = 0$ and

$\mathsf{sslp}_z(t_1) = \mathsf{sslp}_z(t_2) = \mathsf{sslp}_z(t_3) = 0$, thus

$$\begin{aligned}
&\mathsf{elim}^z(t_1 > 0) = (\mathsf{ebreak}) \cup \mathsf{eseg} = \mathsf{breaks}^z_{t_1} \cup \{b + \varepsilon \mid b \in \mathsf{breaks}^z_{t_1}\} = \{-x + \mathbb{Z}\} \cup \{-x + \varepsilon + \mathbb{Z}\}\\
&\mathsf{elim}^z(t_2 \approx 0) = (\mathsf{ebreak}) \cup \mathsf{eseg} = \mathsf{breaks}^z_{t_2} \cup \{b + \varepsilon \mid b \in \mathsf{breaks}^z_{t_2}\} = \{\mathbb{Z}\} \cup \{\varepsilon + \mathbb{Z}\}\\
&\mathsf{elim}^z(t_3 \approx 0) = \{-\infty\}
\end{aligned}$$

. We transform the quantifier into a disjunction and simplify:

$$\neg\exists x, z.\Big(\overbrace{\underbrace{\lceil x+z\rceil - \lfloor x+z\rfloor}_{t_1} > 0 \land \underbrace{\lceil z\rceil - \lfloor z\rfloor}_{t_2} \approx 0 \land \underbrace{\lfloor x\rfloor - x}_{t_3} \approx 0}^{\phi'}\Big)$$

$$\iff \neg\exists x. \begin{pmatrix}
\big(t_1 > 0 \land t_2 \approx 0 \land t_3 \approx 0\big)[\![z /\!/ -\infty]\!]\\
\lor \big(t_1 > 0 \land t_2 \approx 0 \land t_3 \approx 0\big)[\![z /\!/ \mathbb{Z}]\!]\\
\lor \big(t_1 > 0 \land t_2 \approx 0 \land t_3 \approx 0\big)[\![z /\!/ \mathbb{Z} + \varepsilon]\!]\\
\lor \big(t_1 > 0 \land t_2 \approx 0 \land t_3 \approx 0\big)[\![z /\!/ -x + \mathbb{Z}]\!]\\
\lor \big(t_1 > 0 \land t_2 \approx 0 \land t_3 \approx 0\big)[\![z /\!/ -x + \mathbb{Z} + \varepsilon]\!]
\end{pmatrix}$$

$$\iff \neg\exists x. \begin{pmatrix}
\big(t_1 > 0 \land t_2 \approx 0 \land t_3 \approx 0\big)[\![z /\!/ 0]\!]\\
\lor \bigvee_{t \in \mathsf{fin}^{\phi'}_{\mathbb{Z}}} \big(t_1 > 0 \land t_2 \approx 0 \land t_3 \approx 0\big)[\![z /\!/ t]\!]\\
\lor \bigvee_{t \in \mathsf{fin}^{\phi'}_{-x+\mathbb{Z}}} \big(t_1 > 0 \land t_2 \approx 0 \land t_3 \approx 0\big)[\![z /\!/ t]\!]\\
\lor \bigvee_{t \in \mathsf{fin}^{\phi'}_{\mathbb{Z}}} \big(t_1 > 0 \land t_2 \approx 0 \land t_3 \approx 0\big)[\![z /\!/ t + \varepsilon]\!]\\
\lor \bigvee_{t \in \mathsf{fin}^{\phi'}_{-x+\mathbb{Z}}} \big(t_1 > 0 \land t_2 \approx 0 \land t_3 \approx 0\big)[\![z /\!/ t + \varepsilon]\!]
\end{pmatrix} \qquad \begin{aligned}\mathsf{fin}^{\phi'}_{\mathbb{Z}} &= \{0\}\\ \mathsf{fin}^{\phi'}_{-x+\mathbb{Z}} &= \{-x\}\end{aligned}$$

$$\iff \neg\exists x. \begin{pmatrix}
\big(t_1 > 0 \land t_2 \approx 0 \land t_3 \approx 0\big)[\![z /\!/ 0]\!]\\
\lor \big(t_1 > 0 \land t_2 \approx 0 \land t_3 \approx 0\big)[\![z /\!/ 0]\!]\\
\lor \big(t_1 > 0 \land t_2 \approx 0 \land t_3 \approx 0\big)[\![z /\!/ -x]\!]\\
\lor \big(t_1 > 0 \land t_2 \approx 0 \land t_3 \approx 0\big)[\![z /\!/ 0 + \varepsilon]\!]\\
\lor \big(t_1 > 0 \land t_2 \approx 0 \land t_3 \approx 0\big)[\![z /\!/ -x + \varepsilon]\!]
\end{pmatrix}$$

$$\iff \neg\exists x. \begin{pmatrix}
\big(t_1 > 0 \land t_2 \approx 0 \land t_3 \approx 0\big)\ [z/0]\\
\lor \big(t_1 > 0 \land t_2 \approx 0 \land t_3 \approx 0\big)\ [z/-x]\\
\lor \big(\mathsf{lim}^z_{t_1} > 0 \land \mathsf{lim}^z_{t_2} \approx 0 \land \mathsf{lim}^z_{t_3} \approx 0\big)\ [z/0]\\
\lor \big(\underbrace{\mathsf{lim}^z_{t_1}}_{1} > 0 \land \underbrace{\mathsf{lim}^z_{t_2}}_{1} \approx 0 \land \underbrace{\mathsf{lim}^z_{t_3}}_{\lfloor x\rfloor - x} \approx 0\big)\ [z/-x]
\end{pmatrix}$$

$$\iff \neg\exists x. \begin{pmatrix}
\lceil x\rceil - \lfloor x\rfloor > 0 \land 0 \approx 0 \land \lfloor x\rfloor - x \approx 0\\
\lor 1 > 0 \land 1 \approx 0 \land \lfloor x\rfloor - x \approx 0\\
\lor 0 > 0 \land \lceil -x\rceil - \lfloor -x\rfloor \approx 0 \land \lfloor x\rfloor - x \approx 0\\
\lor 1 > 0 \land 1 \approx 0 \land \lfloor x\rfloor - x \approx 0
\end{pmatrix}$$

$$\iff \neg\exists x. \overbrace{(\underbrace{\lceil x\rceil - \lfloor x\rfloor}_{u_1} > 0 \land \underbrace{\lfloor x\rfloor - x}_{u_2} \approx 0)}^{\phi''}$$

Next we compute the elimination set $\mathsf{elim}^x(\phi'') = \mathsf{elim}^x(u_1 > 0) \cup \mathsf{elim}^x(u_2 \approx 0)$:

$$\mathsf{elim}^x(u_1 > 0) = (\mathsf{ebreak}) \cup \mathsf{eseg} = \mathsf{breaks}^x_{u_1} \cup \{b + \varepsilon \mid b \in \mathsf{breaks}^x_{u_1}\} = \{\mathbb{Z}, \mathbb{Z} + \varepsilon\}$$
$$\mathsf{elim}^x(u_2 \approx 0) = (\mathsf{ebreak}) \cup \mathsf{eseg} = \mathsf{breaks}^x_{u_1} \cup \mathsf{ezero} = \{\mathbb{Z}\} \cup \{\mathbb{Z}\}$$

Therefore we can eliminate the quantifier $\exists x$ and simplify:

$$\begin{aligned}
\neg\exists x.\phi'' &\iff \neg\begin{pmatrix} & (u_1 > 0 \wedge u_2 \approx 0)[\![x /\!/ \mathbb{Z}]\!] \\ \vee & (u_1 > 0 \wedge u_2 \approx 0)[\![x /\!/ \mathbb{Z} + \varepsilon]\!] \end{pmatrix} \\
&\iff \neg\begin{pmatrix} & \bigvee_{t \in \mathsf{fin}^{\phi''}_{\mathbb{Z}}} & (u_1 > 0 \wedge u_2 \approx 0)[\![x /\!/ t]\!] \\ \vee & \bigvee_{t \in \mathsf{fin}^{\phi''}_{\mathbb{Z}}} & (u_1 > 0 \wedge u_2 \approx 0)[\![x /\!/ t + \varepsilon]\!] \end{pmatrix} \quad \mathsf{fin}^{\phi''}_{\mathbb{Z}} = \{0\} \\
&\iff \neg\begin{pmatrix} & (u_1 > 0 \wedge u_2 \approx 0)[\![x /\!/ 0]\!] \\ \vee & (u_1 > 0 \wedge u_2 \approx 0)[\![x /\!/ \varepsilon]\!] \end{pmatrix} \\
&\iff \neg\begin{pmatrix} & 0 > 0 \wedge \lfloor 0 \rfloor - 0 \approx 0 \\ \vee & \mathsf{lim}^x_{u_1} > 0[x/0] \wedge \bot \end{pmatrix} \\
&\iff \neg\bot \iff \top
\end{aligned}$$

Thus we established that ϕ is valid.

B Conflict-Driven VIRAS

B.1 CDVS Explanation

CDVS tries to proof or disprove that a conjunction of literals $F = \bigwedge F_i$ where all variables are existentially quantified is valid or unsat. Derivations are preformed on states (F, S, L), starting with $(F, \langle\rangle, \emptyset)$. L is a set of lemmas, that is successively grown and $S = \langle\rangle \mid x_1 \leftarrow \nu_1 \mid \dots \mid x_n \leftarrow \nu_n$ is a list, that represents partial assignment for the variables x_i, with every ν_i being of either a pair $\langle t_i, J_i\rangle$ of a virtual term t_i and a term J_i, $\bot$ or ?, where $\bot$ or ? can only occur as the last element ν_n. An entry in $x \leftarrow \langle t, J\rangle$ is to be understood as "the variable x is virtually substituted by t which is a member of the elimination set of the literal $J \geq 0$". We write F/S for $F[\![x_1 /\!/ t_1]\!] \dots [\![x_n /\!/ t_n]\!]$, if ? and $\bot$ do not occur in S. An entry $x \leftarrow \bot$ can be understood as "no term in $\mathsf{elim}^x(F/S)$ is consistent with our lemmas L/S" and $x \leftarrow ?$ can be understood as "assign some value from the elimination set for x next". The algorithm then starts to add $x \leftarrow \langle t, J\rangle$ for some $t \in \mathsf{elim}^x(F)$ to S as long as choosing these assignments are consistent with the lemmas L. If we arrive in a state where F/S is trivially true and does not contain any variables, we know that F is sat, and the algorithm terminates. This means that for sat instances CDVS does not have to enumerate all elimination sets, but can terminate early in some cases. If in some state F/S is trivially false we found a so-called *leaf conflict*, and if we arrive at a state where $S = S' \mid x_n \leftarrow \bot$, we found a so-called *inner conflict*. Either of these conflicts means that the current assignment S is infeasible, which means the algorithm has to learn a lemma that will prevent it from choosing the same assignment again, then backtracks and continues proof search. If no backtracking can be performed the algorithm returns unsat as this means any assignment makes F false.

B.2 Examples

In order to see why extending the lemma learning to proper virtual terms is not straightforward consider the following example:

$$\phi_{\gtrsim} = x \gtrsim 0 \wedge x \gtrsim 1 \wedge 7 \gtrsim x$$

$$\begin{array}{llll}
 & \langle \phi_{\geq}, & \langle\rangle, & \emptyset\rangle \\
\vdash & \langle \phi_{\geq}, & \langle\rangle \mid x \leftarrow ?, & \emptyset\rangle \\
\vdash & \langle \phi_{\geq}, & \langle\rangle \mid x \leftarrow 0, & \emptyset\rangle \\
\vdash & \langle \phi_{\geq}, & \langle\rangle \mid x \leftarrow 0, & \{x \not\approx 0\}\rangle \\
\vdash & \langle \phi_{\geq}, & \langle\rangle, & \{x \not\approx 0\}\rangle \\
\vdash & \langle \phi_{\geq}, & \langle\rangle \mid x \leftarrow ?, & \{x \not\approx 0\}\rangle \\
\vdash & \langle \phi_{\geq}, & \langle\rangle \mid x \leftarrow 1, & \{x \not\approx 0\}\rangle \\
\vdash & \top & &
\end{array}
\qquad
\begin{array}{llll}
 & \langle \phi_{>}, & \langle\rangle, & \emptyset\rangle \\
\vdash & \langle \phi_{>}, & \langle\rangle \mid x \leftarrow ?, & \emptyset\rangle \\
\vdash & \langle \phi_{>}, & \langle\rangle \mid x \leftarrow \varepsilon, & \emptyset\rangle \\
\vdash & \langle \phi_{>}, & \langle\rangle \mid x \leftarrow \varepsilon, & \{x \leq 0 \vee 1 \leq x\}\rangle \\
\vdash & \langle \phi_{>}, & \langle\rangle, & \{x \leq 0 \vee 1 \leq x\}\rangle \\
\vdash & \langle \phi_{>}, & \langle\rangle \mid x \leftarrow ?, & \{x \leq 0 \vee 1 \leq x\}\rangle \\
\vdash & \langle \phi_{>}, & \langle\rangle \mid x \leftarrow 1+\varepsilon, & \{x \leq 0 \vee 1 \leq x\}\rangle \\
\vdash & \top & &
\end{array}$$

Figure 5: Formal derivations explained in Ex. 16 and 17

Example 16. *Let $\phi_{\geq} = x \geq 0 \wedge x \geq 1 \wedge 7 \geq x$. A formal derivation resulting from this formula is given in Fig. 5. In order to find a value for x the algorithm computes $\mathsf{elim}^x(\phi_{\geq}) = \{0, 1, -\infty\}$ and chooses an arbitrary one of these values, in our case 0. As $x \geq 1[\![x /\!/ 0]\!]$ is trivially false, the algorithm arrives at a leaf conflict, hence it needs to learn a lemma that prevents assigning x to 0. The straight forward way to do this is adding the lemma $x \not\approx 0$. The algorithm then backtracks, continues assigning x to 1 and returns sat.*

Example 17. *Now condsider what happens if consider $\phi_{>}$, the same problem as in Ex. 16 but using strict inequalities (see Fig. 5). We get $\mathsf{elim}^x(\phi_{>}) = \{\varepsilon, 1+\varepsilon, -\infty\}$ and the algorithm chooses to assign x to ε, which again runs into a leaf conflict. In this case one cannot simply learn $x \not\approx \varepsilon$, as this is not a formula in our signature, hence does not have a defined semantics. Intuitively if we want to make sure that x is not assigned to ε we need to make sure that it is assigned to either a value that is smaller than ε, or a value that is greater. Therefore we need to find an upper bound that is greater than ε, but small enough that it does not exclue any other elements of $\mathsf{elim}^x(\phi_{>})$. In our example we can choose 1 and learn the lemma $x \leq 0 \vee 1 \leq x$.*

ε-Lemmas

Example 18. *Consider the formula $\phi = L_1 \wedge L_2$, where $L_1 = \lceil x \rceil - \lfloor x \rfloor - \frac{1}{2} \geq 0$ and $L_2 = x - \lfloor x - \frac{1}{2} \rfloor + 1 > 0$, and we want to find a lemma to exclude an assignment $\langle\rangle \mid x \leftarrow \varepsilon$. In this case we have $\mathsf{nxt}^{\top}_{L_1}(\varepsilon) = \mathsf{nextBreak} = \{1\}$, and $\mathsf{nxt}^{\top}_{L_2}(\varepsilon) = \mathsf{nextBreak}(\varepsilon) \cup \mathsf{curZero}(\varepsilon) = \{\frac{1}{2} + \varepsilon\} \cup \{0\}$. This means that*

$$\begin{aligned}
\mathsf{inFalseInterval}^{\phi}_{\varepsilon}(x) &= 0 < x \wedge \bigwedge_{e \in \mathsf{nxt}^{\top}_{L_1}(\varepsilon)} (\varepsilon \lessdot e \to x \lessdot e) \wedge \bigwedge_{e \in \mathsf{nxt}^{\top}_{L_2}(\varepsilon)} (\varepsilon \lessdot e \to x \lessdot e) \\
&= 0 < x \wedge (\varepsilon \lessdot 1 \to x \lessdot 1) \wedge \big((\varepsilon \lessdot \frac{1}{2} + \varepsilon \to x \lessdot \frac{1}{2} + \varepsilon) \wedge (\varepsilon \lessdot 0 \to x \lessdot 0)\big) \\
&= 0 < x \wedge (0 < 1 \to x < 1) \wedge \big((0 < \frac{1}{2} \to x \leq \frac{1}{2}) \wedge (0 < 0 \to x < 0)\big) \\
&\iff 0 < x \wedge x < 1 \wedge x \leq \frac{1}{2} \iff 0 < x \wedge x \leq \frac{1}{2}
\end{aligned}$$

So we derive the lemma $x \leq 0 \vee \frac{1}{2} < x$.

B.3 Soundness and Completeness

To proof soundness of CDVS in [15] Lemma 2 of [15] asserts that for any derivable state (F, S, L), $(F, S \mid x_{k+1} \leftarrow \bot, L)$, $(F, S \mid x_{k+1} \leftarrow ?, L)$, where $S = \langle x_1 \leftarrow \langle t_1, J_1 \rangle \mid \ldots \mid x_l \leftarrow \langle t_l, J_l \rangle \rangle$ it holds that $\mathbb{R} \models \exists(F/S) \leftrightarrow \exists(F \wedge \bigwedge_{i=1}^{l} J_i \approx 0)$. Lemma 2 of [15] is then used to proof Lemma 3 of [15] which is in turn needed for soundness of the calculus. Inspecting the proof of Lemma 4 [15] we can see that only a weaker form of Lemma 2 [15] is needed: $\forall\neg(F/S) \to \forall(F \to \bigvee_{i=1}^{l} J_i \not\approx 0)$. In the case of CDVS $J_i \not\approx 0$, are the lemmas that will be derived for excluding solutions $x_i \neq t_i$. If we replace this by $\mathsf{lemma}_F(x_i \not\approx t_i)$ we get the following alternative invariant.

Lemma 8. *For any state* (F, S, L), $(F, S \mid x_{k+1} \leftarrow \bot, L)$, *or* $(F, S \mid x_{k+1} \leftarrow ?, L)$, *where* $S = \langle x_1 \leftarrow \langle t_1, J_1 \rangle \mid \ldots \mid x_k \leftarrow \langle t_k, J_k \rangle \rangle$

$$\mathbb{R} \models \forall\neg(F/S) \to \forall(F \to \bigvee_{i=1}^{k} \mathsf{lemma}_F(x_i \not\approx t_i))$$

Proof. Apply induction on k, and use Lem. 7.1. □

By using Lem. 8 we can proof Lemma 3 from [15], and therefore soundness in the same way as in [15].

For showing completeness a well-founded ordering on states (F, S, L) is defined in [15], which is based on a lexicographical ordering with a component n, the number of so-called active nodes in the search tree. Active nodes are the nodes that are not deactivated L Due to Lem. 7.2 we know that our modified rules will reduce the number of active nodes as in CDVS, which means that CD-Viras is complete as well.

C Definitions

Definition 1 (Slope and Period; repeated)**.** *Let t be a* Lira *term. By recursion on t, we define the **period** per_t^x, **outer slope** oslp_t^x, and **segment slope** sslp_t^x of t as:*

$$\mathsf{oslp}_y^x = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases} \quad \mathsf{oslp}_1^x = 0 \quad \mathsf{oslp}_{kt}^x = k \cdot \mathsf{oslp}_t^x \quad \mathsf{oslp}_{s+t}^x = \mathsf{oslp}_s^x + \mathsf{oslp}_t^x \quad \mathsf{oslp}_{\lfloor t \rfloor}^x = \mathsf{oslp}_t^x$$

$$\mathsf{sslp}_y^x = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases} \quad \mathsf{sslp}_1^x = 0 \quad \mathsf{sslp}_{kt}^x = k \cdot \mathsf{sslp}_t^x \quad \mathsf{sslp}_{s+t}^x = \mathsf{sslp}_s^x + \mathsf{sslp}_t^x \quad \mathsf{sslp}_{\lfloor t \rfloor}^x = 0$$

$$\mathsf{per}_y^x = \mathsf{per}_1^x = 0 \qquad \mathsf{per}_{kt}^x = \mathsf{per}_t^x$$

$$\mathsf{per}_{s+t}^x = \begin{cases} \mathsf{per}_s^x & \text{if } \mathsf{per}_t^x = 0 \\ \mathsf{per}_t^x & \text{if } \mathsf{per}_s^x = 0 \\ \mathsf{lcm}^{\mathbb{Q}}\{\mathsf{per}_s^x, \mathsf{per}_t^x\} & \text{otherwise} \end{cases} \qquad \mathsf{per}_{\lfloor t \rfloor}^x = \begin{cases} 0 & \text{if } \mathsf{per}_t^x = 0 = \mathsf{oslp}_t^x \\ \frac{1}{|\mathsf{oslp}_t^x|} & \text{if } \mathsf{per}_t^x = 0 \neq \mathsf{oslp}_t^x \\ \mathsf{num}(\mathsf{per}_t^x) \cdot \mathsf{den}(\mathsf{oslp}_t^x) & \text{otherwise} \end{cases}$$

Definition 13 ($\mathbb{Q}$-lcm)**.** *Let Q be a finite subset of $\mathbb{Q}$ such that $0 \notin \mathbb{Q}$. We define*

$$\mathsf{lcm}^{\mathbb{Q}}(Q) = \frac{lcm\{\mathsf{num}(q) \mid q \in Q\}}{gcd\{\mathsf{den}(q) \mid q \in Q\}}$$

Definition 14 (Quotient Remainder)**.** *For any* Lira*-term t and $q \in \mathbb{Q}$ we define the gerneralized **quotient** and **remainder** of t and q:*

$$\mathsf{quot}_q(t) = \left\lfloor \frac{t}{q} \right\rfloor \qquad \mathsf{rem}_q(t) = t - q \cdot \mathsf{quot}_q(t)$$

Definition 2 (Bound Distance; repeated)**.** *Let t be a* LIRA*-term. We define* $\mathsf{distY}^{\pm}_{x,t} \in \mathbf{T}$ *and* $\mathsf{deltaY}_{x,t} \in \mathbb{Q}$ *by recursion on t:*

$$\begin{aligned}
\mathsf{deltaY}_{x,y} &= 0 \\
\mathsf{deltaY}_{x,1} &= 0 \\
\mathsf{deltaY}_{x,kt} &= |k|\mathsf{deltaY}_{x,t} \\
\mathsf{deltaY}_{x,s+t} &= \mathsf{deltaY}_{x,s} + \mathsf{deltaY}_{x,t} \\
\mathsf{deltaY}_{x,\lfloor t \rfloor} &= \mathsf{deltaY}_{x,t} + 1 \\
\mathsf{distY}^{+}_{x,t} &= \mathsf{distY}^{-}_{x,t} + \mathsf{deltaY}_{x,t}
\end{aligned}
\qquad
\begin{aligned}
\mathsf{distY}^{-}_{x,y} &= \begin{cases} 0 & \text{if } x = y \\ y & \text{otherwise} \end{cases} \\
\mathsf{distY}^{-}_{x,1} &= 1 \\
\mathsf{distY}^{-}_{x,kt} &= \begin{cases} k \cdot \mathsf{distY}^{-}_{x,t} & \text{if } k \geq 0 \\ k \cdot \mathsf{distY}^{+}_{x,t} & \text{if } k < 0 \end{cases} \\
\mathsf{distY}^{-}_{x,s+t} &= \mathsf{distY}^{-}_{x,s} + \mathsf{distY}^{-}_{x,t} \\
\mathsf{distY}^{-}_{x,\lfloor t \rfloor} &= \mathsf{distY}^{-}_{x,t} - 1
\end{aligned}$$

Definition 3 (Limit; repeated)**.** *The* ***limit term*** lim^x_t *of a* LIRA*-term t wrt x is defined by recursion on t, as:*

$$\begin{aligned}
\mathsf{lim}^x_y &= y \\
\mathsf{lim}^x_1 &= 1 \\
\mathsf{lim}^x_{kt} &= k \cdot \mathsf{lim}^x_t
\end{aligned}
\qquad
\begin{aligned}
\mathsf{lim}^x_{s+t} &= \mathsf{lim}^x_s + \mathsf{lim}^x_t \\
\mathsf{lim}^x_{\lfloor t \rfloor} &= \begin{cases} \lfloor \mathsf{lim}^x_t \rfloor & \text{if } \mathsf{sslp}^x_t \geq 0 \\ \lceil \mathsf{lim}^x_t \rceil - 1 & \text{if } \mathsf{sslp}^x_t < 0 \end{cases}
\end{aligned}$$

We write lim_t *for* lim^x_t *if x is clear in the context.*

Definition 4 (Segment Line; repeated)**.** *The* ***segment distance*** $\mathsf{dseg}_t(x_0)$ *of a* LIRA*-term t at x_0 is:*

$$\mathsf{dseg}^x_t(x_0) = -\mathsf{sslp}^x_t \cdot x_0 + \mathsf{lim}^x_t[x_0] \qquad \mathsf{zero}^x_t(x_0) = x_0 - \frac{\mathsf{lim}^x_t[x_0]}{\mathsf{sslp}_t}$$

The ***segment line*** *of t at x_0 is* $\mathsf{sslp}_t \cdot x + \mathsf{dseg}_t(x_0)$*, whereas* $\mathsf{zero}_t(x_0)$ *is the* ***zero of the segment*** *of t at x_0.*

Definition 5 (Grid Intersection; repeated)**.** *For $s, t \in \mathbf{T}$ and $p, k \in \mathbb{Q}^{>0}$, the* ***grid intersection*** *is*

$$(s + p\mathbb{Z}) \sqcap (\!|t, t+k|\!) = \{start_{(\!|} + np \mid n \in \mathbb{N}, np \lessdot_{|\!)} k\}$$

where

$$\begin{aligned}
\lceil t \rceil^{s+p\mathbb{Z}} &= t + \mathsf{rem}_p(s-t) & \lceil t+\varepsilon \rceil^{s+p\mathbb{Z}} &= \lfloor t+p \rfloor^{s+p\mathbb{Z}} & start_{[} &= \lceil t \rceil^{s+p\mathbb{Z}} & \lessdot_{]} &= \leq \\
\lfloor t \rfloor^{s+p\mathbb{Z}} &= t - \mathsf{rem}_p(t-s) & \lfloor t-\varepsilon \rfloor^{s+p\mathbb{Z}} &= \lceil t-p \rceil^{s+p\mathbb{Z}} & start_{(} &= \lceil t+\varepsilon \rceil^{s+p\mathbb{Z}} & \lessdot_{)} &= <
\end{aligned}$$

Definition 6 (repeated)**.** *The set of* ***discontinuities*** breaks^x_t *of a* LIRA*-term t wrt variable x is defined by recursion on t, as:*

$$\begin{aligned}
\mathsf{breaks}^x_y &= \mathsf{breaks}^x_1 = \emptyset \\
\mathsf{breaks}^x_{kt} &= \mathsf{breaks}^x_t \\
\mathsf{breaks}^x_{s+t} &= \mathsf{breaks}^x_s \cup \mathsf{breaks}^x_t
\end{aligned}
\qquad
\mathsf{breaks}^x_{\lfloor t \rfloor} = \begin{cases} \mathsf{breaks}^x_t & \text{if } \mathsf{sslp}_t = 0 \\ \{\mathsf{zero}_t(0) + \mathsf{per}_{\lfloor t \rfloor}\mathbb{Z}\} & \text{if } \mathsf{breaks}^x_t = \emptyset \;\&\; \mathsf{sslp}_t \neq 0 \\ \mathsf{breaks}^x_t \cup \mathsf{breaksInSeg}^x_t & \text{if } \mathsf{breaks}^x_t \neq \emptyset \;\&\; \mathsf{sslp}_t \neq 0 \end{cases}$$

$$\mathsf{breaksInSeg}^x_t = \left\{ b + \mathsf{per}_{\lfloor t \rfloor}\mathbb{Z} \;\middle|\; \begin{array}{ll} b \in (\mathsf{zero}(b_0) + \frac{1}{\mathsf{sslp}_t}\mathbb{Z}) \sqcap [b_0, b_0 + p^{\min}_t) & \text{where} \\ b_0 \in (b'_0 + p\mathbb{Z}) \sqcap [b'_0, b'_0 + \mathsf{per}_{\lfloor t \rfloor}) & \text{where} \\ b'_0 + p\mathbb{Z} \in \mathsf{breaks}_t & \end{array} \right\}$$

$$\begin{aligned}
p^{\min}_t &= \min\{p \mid b + p\mathbb{Z} \in \mathsf{breaks}^x_t\} \\
\mathsf{breaks}^{x,\infty}_t &= \{t + pz \mid z \in \mathbb{Z}, t + p\mathbb{Z} \in \mathsf{breaks}^x_t\}
\end{aligned}$$

Definition 15 (Periodic Literal). *Let t be a* LIRA *term. We call a* LIRA*-term* ***periodic*** *if* $\mathsf{oslp}_t = 0$ *and* ***aperiodic*** *otherwise. We extend this definition to* LIRA*-literals* $t \diamond 0$.

Definition 7 (Core Interval; repeated). *Let t be a* LIRA*-term with* $\mathsf{oslp}_t \neq 0$. *The* ***core interval*** *of t is* $[\mathsf{distX}_t^-, \mathsf{distX}_t^+]$, *where*

$$\mathsf{distX}_t^- = -\frac{\mathsf{distY}_t^{\mathsf{sgn}(\mathsf{oslp}_t)}}{\mathsf{oslp}_t} \qquad \mathsf{deltaX}_t = \frac{\mathsf{deltaY}_t}{|\mathsf{oslp}_t|} \qquad \mathsf{distX}_t^+ = \mathsf{distX}_t^- + \mathsf{deltaX}_t$$

Definition 8 (Virtual Term; repeated). *A* ***virtual term*** *v is a sum $t + e\varepsilon + z\mathbb{Z} + i\infty$ with $t \in \mathbf{T}, e \in \{0,1\}, z \in \mathbb{Q}^{\geq 0}, i \in \{0,+,-\}$, where $z = 0$ or $i = 0$. We may omit summands with zero coefficients. We write $\mathbb{Z}(v) = z$, $\varepsilon(v) = e$ and $\infty(v) = i$. A virtual term is* ***plain*** *if $e = z = i = 0$ and* ***proper*** *otherwise.*

Definition 9 (Virtual Substitution; repeated). *A* ***virtual substitution function*** $\circ[\![\circ \mathbin{/\!/} \circ]\!]$ *maps a conjunction of* LIRA*-literals, a variable, and a virtual term to a formula. We write $\phi[\![t]\!]$ for $\phi[\![x \mathbin{/\!/} t]\!]$. Let ϕ be a conjunction of* LIRA*-literals, t a term, v a virtual term with $\mathbb{Z}(v) = 0$, $P = \{L \in \phi \mid L \text{ is periodic }\}$, and $A = \{L \in \phi \mid L \text{ is aperiodic }\}$. Then,*

1. $\phi[\![x \mathbin{/\!/} t + e\varepsilon + p\mathbb{Z}]\!] = \bigvee_{t' \in \mathsf{fin}^{\phi}_{t+p\mathbb{Z}}} \phi[\![x \mathbin{/\!/} t' + e\varepsilon]\!]$ *where*

   V1. if $\forall L \in A. \mathsf{lim}_L^{\pm\infty} = \top$: $\quad \mathsf{fin}^{\phi}_{t+p\mathbb{Z}} = \{s \pm \infty \mid s \in \big(t + p\mathbb{Z} \sqcap [t, t+\lambda)\big) \}$

   V2. if $\exists L \in A. L = u \approx 0$: $\quad \mathsf{fin}^{\phi}_{t+p\mathbb{Z}} = \big(t + p\mathbb{Z} \sqcap [\mathsf{distX}^-_{u\approx 0}, \mathsf{distX}^+_{u\approx 0}]\big)$

   V3. otherwise: $\quad \mathsf{fin}^{\phi}_{t+p\mathbb{Z}} = \bigcup_{L \in A, \mathsf{lim}_L^{-\infty} = \bot} \big(t + p\mathbb{Z} \sqcap [\mathsf{distX}^-_L, \mathsf{distX}^+_L + \lambda]\big)$

   $\lambda = \mathsf{lcm}^{\mathbb{Q}}(\{p\} \cup \{\mathsf{per}_L \mid L \in P\})$

2. $(\bigwedge_{L\in\phi} L)[\![x \mathbin{/\!/} v]\!] = \bigwedge_{L\in\phi} (L[\![x \mathbin{/\!/} v]\!])$

3. $(s \diamond 0)[\![x \mathbin{/\!/} v \pm \infty]\!] = \begin{cases} \mathsf{lim}^{\pm\infty}_{s\diamond 0} & \text{if } s \diamond 0 \text{ is aperiodic } (\mathsf{oslp}_s = 0) \\ (s \diamond 0)[\![x \mathbin{/\!/} v]\!] & \text{if } s \diamond 0 \text{ is periodic } (\mathsf{oslp}_s \neq 0) \end{cases}$

4. $((\neg)s \approx 0)[\![x \mathbin{/\!/} t + \varepsilon]\!] = \begin{cases} (\neg)\bot & \text{if } \mathsf{sslp}_s \neq 0 \\ (\neg)\mathsf{lim}_s[t] \approx 0 & \text{if } \mathsf{sslp}_s = 0 \end{cases}$

5. $(s \gtrsim 0)[\![x \mathbin{/\!/} t + \varepsilon]\!] = \begin{cases} \mathsf{lim}_s[t] \geq 0 & \text{if } \mathsf{sslp}_s > 0 \\ \mathsf{lim}_s[t] \gtrsim 0 & \text{if } \mathsf{sslp}_s = 0 \\ \mathsf{lim}_s[t] > 0 & \text{if } \mathsf{sslp}_s < 0 \end{cases}$

6. $(s \diamond 0)[\![x \mathbin{/\!/} t]\!] = s[x/t] \diamond 0$

Definition 10 (Elimination Set; repeated). *The* ***elimination set*** $\mathsf{elim}^x(\phi)$ *of a conjunction of literals ϕ with respect to the variable x is defined in Fig. 2.*

Definition 16 (Solution Interval)**.** *Let ϕ be a* LIRA*-formula and t a virtual term. We define the formula* sInter*, and say that $[t, x]$ is a solution interval of ϕ with respect to some $\mathbb{R}$-interpretation $\mathcal{I}$ if $\mathcal{I} \models \mathsf{sInter}(t, x, \phi)$.*

$$\mathsf{sInter}(t, x, \phi) = \begin{cases} t \dot{\leq} x \wedge \forall x'(t \dot{\leq} x' \leq x \to \phi[x']) & \text{if } \mathbb{Z}(t) = 0 \\ \exists z \in \mathbb{Z}.\mathsf{sInter}(t' + pz) & \text{if } t = t' + p\mathbb{Z} \end{cases}$$

where

$$\begin{aligned} t \dot{\leq} x &= t \leq x & \qquad t - \infty \dot{\leq} x &= \top \\ t + \varepsilon \dot{\leq} x &= t < x & \qquad t + \infty \dot{\leq} x &= \bot \end{aligned}$$

Definition 11 (False Interval; repeated)**.** *Let ϕ be a conjunction of literals. The **false interval** of ϕ at $t + \varepsilon$ is denoted as $\mathsf{inFalseInterval}^{\phi}_{t+\varepsilon}(x)$ and defined in Fig. 3.*

Definition 12 (CD-VIRAS Lemmas; repeated)**.** *Let ϕ be a conjunction of literals, t a term, and $A = \{L \mid L \in \phi, \mathsf{oslp}_L \neq 0\}$. The **lemma function** of* CD-VIRAS *is defined as: conflicts are:*

$$\begin{aligned}
&\mathsf{lemma}_\phi\ (x \not\approx t) &&= x \not\approx t \\
&\mathsf{lemma}_\phi\ (x \not\approx t + \varepsilon) &&= \neg\mathsf{inFalseInterval}^{\phi}_{t+\varepsilon}(x) \\
&\mathsf{lemma}_\phi\ (x \not\approx t + e\varepsilon + \infty) &&= x \leq \mathsf{distX}^{+}_{L} && \text{if } \mathsf{lim}^{+\infty}_{L} = \bot \text{ for some } L \in A \\
&\mathsf{lemma}_\phi\ (x \not\approx t + e\varepsilon - \infty) &&= \mathsf{distX}^{-}_{L} \leq x && \text{if } \mathsf{lim}^{-\infty}_{L} = \bot \text{ for some } L \in A \\
&\mathsf{lemma}_\phi\ (x \not\approx t \pm \infty) &&= \mathsf{rem}_\lambda(x) \not\approx \mathsf{rem}_\lambda(t) \\
&\mathsf{lemma}_\phi\ (x \not\approx t + \varepsilon \pm \infty) &&= \neg\mathsf{inFalseInterval}^{\phi}_{t+\lambda(\mathsf{quot}_\lambda(x) - \mathsf{quot}_\lambda(t))+\varepsilon}(x)
\end{aligned}$$

D Proofs

Lemma 9 ($\mathbb{Q}$-lcm)**.** *If Q is a finite subset of $\mathbb{Q}$ such that $0 \notin \mathbb{Q}$, then $q \in Q \Longrightarrow \frac{\mathsf{lcm}^{\mathbb{Q}}(Q)}{q} \in \mathbb{Z}$*

Proof. Follows straight from the defintion $\mathsf{lcm}^{\mathbb{Q}}(Q) = \frac{lcm\{\mathsf{num}(q) | q \in Q\}}{gcd\{\mathsf{den}(q) | q \in Q\}}$. □

Lemma 10 (Quotient Remainder)**.** *For $0 \in \mathbb{Q}^{>0}$ we have*

1. $\mathbb{R} \models x \approx p \cdot \mathsf{quot}_p(x) + \mathsf{rem}_p(x)$
2. $\mathbb{R} \models 0 \leq \mathsf{rem}_p(x) < p$
3. $\mathbb{R} \models \mathsf{quot}_p(x) \in \mathbb{Z}$

Proof. 1.

$$\begin{aligned} p \cdot \mathsf{quot}_p(x) + \mathsf{rem}_p(x) &= p \cdot \mathsf{quot}_p(x) + x - p \cdot \mathsf{quot}_p(x) \\ &\approx x \end{aligned}$$

2.

$$\begin{aligned} & 0 \leq \mathsf{rem}_p(x) < p \\ \Longleftrightarrow & 0 \leq x - p \left\lfloor \tfrac{x}{p} \right\rfloor < p \\ \Longleftrightarrow & 0 \leq \tfrac{x}{p} - \left\lfloor \tfrac{x}{p} \right\rfloor < 1 \end{aligned}$$

Which obviously holds.

3. By defintion as $\mathsf{quot}_q(x) = \left\lfloor \frac{x}{q} \right\rfloor$.

□

Lemma 11 (Linear Term)**.** *Let t be* Lira*-term, with* $\mathsf{per} = 0$*, or* $\mathsf{breaks} = \emptyset$*. Then we say the term is a **linear term**. For all linear terms we have* $\mathsf{per} = 0$*,* $\mathsf{breaks} = \emptyset$*,* $\mathsf{oslp} = \mathsf{sslp}$*,* $\mathbb{R} \models \mathsf{lim}_t \approx t$*,* $\mathbb{R} \models t[x] \approx \mathsf{oslp} \cdot x + t[0]$*, and* $\mathbb{R} \models \forall y. t[x] \approx \mathsf{oslp} \cdot (x - y) + t[y]$*.*

Proof. Everything but the last property can be shown straight forward by induction on t. The last one follows from the other ones:

$$\begin{aligned} t[x] \approx & t[x] + \mathsf{oslp} \cdot y - \mathsf{oslp} \cdot y \\ \approx & \mathsf{oslp} \cdot x + t[0] + \mathsf{oslp} \cdot y - \mathsf{oslp} \cdot y \\ \approx & \mathsf{oslp} \cdot (x - y) + \underbrace{\mathsf{oslp} \cdot y + t[0]}_{t[y]} \end{aligned}$$

□

Lemma 1 (Periodic Shift; repeated)**.** *If* $\mathsf{per}_t \neq 0$ *then* $\mathbb{R} \models \forall x, y. \big(t[x + \mathsf{per} \lfloor y \rfloor] \approx t[x] + \mathsf{oslp} \cdot \mathsf{per} \lfloor y \rfloor \big)$

Proof. We apply induction on t.

base case $t = v \in \mathbf{V} \cup \{1\}$**:**

Then $\mathsf{per}_v = 0$, thus we get $t[x + \underbrace{\mathsf{per}_v}_{0} \lfloor y \rfloor] \approx t[x] + \mathsf{oslp}_v \cdot \underbrace{\mathsf{per}_v}_{0} \lfloor y \rfloor$

⊣

base case $t = v \in \mathbf{V} \cup \{1\}$.

inductive case $t = t_0 + t_1$**:**

case $\mathsf{per}_{t_0} = 0$**:**

Then we have $\mathsf{per}_{t_0+t_1} = \mathsf{per}_{t_1}$.

$$\begin{aligned} & (t_0 + t_1)[x + \mathsf{per}_{t_0+t_1} \lfloor y \rfloor] \\ \approx\ & t_0[x + \mathsf{per}_{t_1} \lfloor y \rfloor] + t_1[x + \mathsf{per}_{t_1} \lfloor y \rfloor] \\ \approx\ & t_0[x + \mathsf{per}_{t_1} \lfloor y \rfloor] + t_1[x] + \mathsf{oslp}_{t_1} \cdot \mathsf{per}_{t_1} \lfloor y \rfloor && \text{by I.H.} \\ \approx\ & t_0[x] + \mathsf{oslp}_{t_0} \cdot \mathsf{per}_{t_1} \lfloor y \rfloor + t_1[x] + \mathsf{oslp}_{t_1} \cdot \mathsf{per}_{t_1} \lfloor y \rfloor && \text{by Linear Term (Lem. 11)} \\ \approx\ & t_0[x] + \mathsf{oslp}_{t_0} \cdot \mathsf{per}_{t_0+t_1} \lfloor y \rfloor + t_1[x] + \mathsf{oslp}_{t_1} \cdot \mathsf{per}_{t_0+t_1} \lfloor y \rfloor \\ \approx\ & (t_0 + t_1)[x] + \underbrace{(\mathsf{oslp}_{t_0} + \mathsf{oslp}_{t_1})}_{\mathsf{oslp}_{t_0+t_1}} \cdot \mathsf{per}_{t_0+t_1} \lfloor y \rfloor \end{aligned}$$

⊣

case $\mathsf{per}_{t_0} = 0$.

case $\mathsf{per}_{t_1} = 0$**:**

Same as the case before.

⊣

case $\mathsf{per}_{t_1} = 0$.

case $\mathsf{per}_{t_0} \neq 0 \neq \mathsf{per}_{t_1}$**:**

In this case we have $\mathsf{per}_{t_0+t_1} \approx \mathsf{lcm}^{\mathbb{Q}}\{\mathsf{per}_{t_0}, \mathsf{per}_{t_1}\}$. Notice that by $\mathbb{Q}$-lcm (Lem. 9) we know that (P_1) this means by $\mathbb{Q}$-lcm (Lem. 9) that $\frac{\mathsf{per}_{t_0+t_1}}{\mathsf{per}_{t_0}}, \frac{\mathsf{per}_{t_0+t_1}}{\mathsf{per}_{t_1}} \in \mathbb{Z}$.

$$
\begin{aligned}
& (t_0+t_1)[x + \mathsf{per}_{t_0+t_1} \lfloor y \rfloor\,] \\
\approx\ & t_0[x + \mathsf{per}_{t_0+t_1} \lfloor y \rfloor\,] + t_1[x + \mathsf{per}_{t_0+t_1} \lfloor y \rfloor\,] \\
\approx\ & t_0[x + \mathsf{per}_{t_0} \cdot \tfrac{\mathsf{per}_{t_0+t_1}}{\mathsf{per}_{t_0}} \lfloor y \rfloor\,] + t_1[x + \mathsf{per}_{t_1} \cdot \tfrac{\mathsf{per}_{t_0+t_1}}{\mathsf{per}_{t_1}} \lfloor y \rfloor\,] \\
\approx\ & t_0[x + \mathsf{per}_{t_0} \cdot \left\lfloor \tfrac{\mathsf{per}_{t_0+t_1}}{\mathsf{per}_{t_0}} \lfloor y \rfloor \right\rfloor\,] + t_1[x + \mathsf{per}_{t_1} \cdot \left\lfloor \tfrac{\mathsf{per}_{t_0+t_1}}{\mathsf{per}_{t_1}} \lfloor y \rfloor \right\rfloor\,] \\
\approx\ & t_0[x] + \mathsf{oslp}_{t_0}\mathsf{per}_{t_0} \cdot \left\lfloor \tfrac{\mathsf{per}_{t_0+t_1}}{\mathsf{per}_{t_0}} \lfloor y \rfloor \right\rfloor + t_1[x] + \mathsf{oslp}_{t_1}\mathsf{per}_{t_1} \cdot \left\lfloor \tfrac{\mathsf{per}_{t_0+t_1}}{\mathsf{per}_{t_1}} \lfloor y \rfloor \right\rfloor \text{ by I.H.} \\
\approx\ & t_0[x] + \mathsf{oslp}_{t_0} \cdot \mathsf{per}_{t_0+t_1} \lfloor y \rfloor + t_1[x] + \mathsf{oslp}_{t_1} \cdot \mathsf{per}_{t_0+t_1} \lfloor y \rfloor \\
\approx\ & (t_0+t_1)[x] + \underbrace{(\mathsf{oslp}_{t_0} + \mathsf{oslp}_{t_1})}_{\mathsf{oslp}_{t_0+t_1}} \cdot \mathsf{per}_{t_0+t_1} \lfloor y \rfloor
\end{aligned}
$$

⊣

case $\mathsf{per}_{t_0} \neq 0 \neq \mathsf{per}_{t_1}$.

inductive case $t = t_0 + t_1$.

inductive case $t = kt_0$**:**

$$
\begin{aligned}
& (kt_0)[x + \mathsf{per}_{kt_0} \lfloor y \rfloor\,] \\
\approx\ & (kt_0)[x + \mathsf{per}_{t_0} \lfloor y \rfloor\,] \\
\approx\ & k(t_0[x + \mathsf{per}_{t_0} \lfloor y \rfloor\,]) \\
\approx\ & k(t_0[x] + \mathsf{oslp}_{t_0}\mathsf{per}_{t_0} \lfloor y \rfloor) \\
\approx\ & kt_0[x] + \underbrace{k\mathsf{oslp}_{t_0}}_{\mathsf{oslp}_{kt_0}} \cdot \mathsf{per}_{t_0} \lfloor y \rfloor
\end{aligned}
$$

⊣

inductive case $t = kt_0$.

inductive case $t = \lfloor t_0 \rfloor$**:**

case $\mathsf{per}_{t_0} = 0$ & $\mathsf{oslp}_{t_0} = 0$**:**

Then $\mathsf{per}_{\lfloor t_0 \rfloor} = 0$ hence the hypothesis is not applicable.

⊣

case $\mathsf{per}_{t_0} = 0$ & $\mathsf{oslp}_{t_0} = 0$.

case $\mathsf{per}_{t_0} = 0$ & $\mathsf{oslp} \neq 0$**:**

Then by Linear Term (Lem. 11).

$$\begin{aligned}
&\lfloor t_0[x + \tfrac{1}{|\mathsf{oslp}_{t_0}|} \lfloor y \rfloor] \rfloor \\
\approx\ &\lfloor t_0[x] + \underbrace{\mathsf{oslp}_{t_0} \tfrac{1}{|\mathsf{oslp}_{t_0}|}}_{\in \mathbb{Z}} \lfloor y \rfloor \rfloor \quad \text{by Linear Term (Lem. 11)} \\
\approx\ &\lfloor t_0[x] \rfloor + \underbrace{\mathsf{oslp}_{t_0}}_{\mathsf{oslp}_{\lfloor t_0 \rfloor}} \underbrace{\tfrac{1}{|\mathsf{oslp}_{t_0}|}}_{\mathsf{per}_{\lfloor t_0 \rfloor}} \lfloor y \rfloor
\end{aligned}$$

⊣

case $\mathsf{per}_{t_0} = 0$ & $\mathsf{oslp} \neq 0$.

case $\mathsf{per}_{t_0} \neq 0$:

$$\begin{aligned}
&\left\lfloor t_0[x + \mathsf{per}_{\lfloor t_0 \rfloor} \lfloor y \rfloor] \right\rfloor \\
\approx\ &\lfloor t_0[x + \mathsf{num}(\mathsf{per}_{t_0})\mathsf{den}(\mathsf{oslp}_{t_0}) \lfloor y \rfloor] \rfloor \\
\approx\ &\left\lfloor t_0[x + \mathsf{den}(\mathsf{oslp}_{t_0}) \tfrac{\mathsf{num}(\mathsf{per}_{t_0}) \cdot \mathsf{den}(\mathsf{per}_{t_0})}{\mathsf{den}(\mathsf{per}_{t_0})} \lfloor y \rfloor] \right\rfloor \\
\approx\ &\lfloor t_0[x + \mathsf{per}_{t_0} \lfloor \mathsf{den}(\mathsf{oslp}_{t_0})\mathsf{den}(\mathsf{per}_{t_0}) \lfloor y \rfloor \rfloor] \rfloor \\
\approx\ &\lfloor t_0[x] + \mathsf{oslp}_{t_0}\mathsf{per}_{t_0} \lfloor \mathsf{den}(\mathsf{oslp}_{t_0})\mathsf{den}(\mathsf{per}_{t_0}) y \rfloor \rfloor \quad \text{by I.H.} \\
\approx\ &\lfloor t_0[x] + \underbrace{\mathsf{oslp}_{t_0}\mathsf{den}(\mathsf{oslp}_{t_0})}_{\in \mathbb{Z}} \underbrace{\mathsf{per}_{t_0}\mathsf{den}(\mathsf{per}_{t_0})}_{\in \mathbb{Z}} \lfloor y \rfloor \rfloor \\
\approx\ &\lfloor t_0[x] \rfloor + \mathsf{oslp}_{t_0}\mathsf{den}(\mathsf{oslp}_{t_0})\mathsf{per}_{t_0}\mathsf{den}(\mathsf{per}_{t_0}) \lfloor y \rfloor \\
\approx\ &\lfloor t_0[x] \rfloor + \underbrace{\mathsf{oslp}_{t_0}}_{\mathsf{oslp}_{\lfloor t_0 \rfloor}} \underbrace{\mathsf{den}(\mathsf{oslp}_{t_0})\mathsf{num}(\mathsf{per}_{t_0})}_{\mathsf{per}_{\lfloor t_0 \rfloor}} \lfloor y \rfloor
\end{aligned}$$

⊣

case $\mathsf{per}_{t_0} \neq 0$.

inductive case $t = \lfloor t_0 \rfloor$.

□

Lemma 2 (Linear Bounds; repeated)**.** $\mathbb{R} \models \forall x.\Big(\mathsf{oslp} \cdot x + \mathsf{distY}^- \leq t \leq \mathsf{oslp} \cdot x + \mathsf{distY}^+\Big)$.

Proof. We apply induction on t.

base case $t = 1$**:**

$$\underbrace{\mathsf{oslp}_1}_{0} \cdot x + \underbrace{\mathsf{distY}^-_1}_{1} \leq 1 \leq \underbrace{\mathsf{oslp}_1}_{0} \cdot x + \underbrace{\mathsf{distY}^-_1}_{1} + \underbrace{\mathsf{deltaY}_1}_{0}$$

⊣

base case $t = 1$.

base case $t = y \in \mathbf{V}$ & $x = y$**:**

$$\underbrace{\mathsf{oslp}_y}_{1} \cdot x + \underbrace{\mathsf{distY}^-_y}_{0} \leq \underbrace{y}_{x} \leq \underbrace{\mathsf{oslp}_y}_{1} \cdot x + \underbrace{\mathsf{distY}^-_y}_{0} + \underbrace{\mathsf{deltaY}_y}_{0}$$

⊣

base case $t = y \in \mathbf{V}$ & $x = y$.

base case $t = y \in \mathbf{V}$ & $x \neq y$:

$$\underbrace{\mathsf{oslp}_y}_{0} \cdot x + \underbrace{\mathsf{distY}^-_y}_{y} \leq y \leq \underbrace{\mathsf{oslp}_y}_{0} \cdot x + \underbrace{\mathsf{distY}^-_y}_{y} + \underbrace{\mathsf{deltaY}_y}_{0}$$

⊣

base case $t = y \in \mathbf{V}$ & $x \neq y$.

inductive case $t = t_0 + t_1$:

By I.H. we have

$$\mathsf{oslp}_{t_0} \cdot x + \mathsf{distY}^-_{t_0} \leq t_0 \leq \mathsf{oslp}_{t_0} \cdot x + \mathsf{distY}^-_{t_0} + \mathsf{deltaY}_{t_0}$$
$$\mathsf{oslp}_{t_1} \cdot x + \mathsf{distY}^-_{t_1} \leq t_1 \leq \mathsf{oslp}_{t_1} \cdot x + \mathsf{distY}^-_{t_1} + \mathsf{deltaY}_{t_1}$$

Thus we have

$$\mathsf{oslp}_{t_0} x + \mathsf{distY}^-_{t_0} + \mathsf{oslp}_{t_1} x + \mathsf{distY}^-_{t_1} \leq t_0 + t_1$$
$$\iff (\underbrace{\mathsf{oslp}_{t_0} + \mathsf{oslp}_{t_1}}_{\mathsf{oslp}_{t_0+t_1}})x + \underbrace{\mathsf{distY}^-_{t_0} + \mathsf{distY}^-_{t_1}}_{\mathsf{distY}^-_{t_0+t_1}} \leq t_0 + t_1$$

and

$$t_0 + t_1 \leq \mathsf{oslp}_{t_0} x + \mathsf{distY}^-_{t_0} + \mathsf{deltaY}_{t_0} + \mathsf{oslp}_{t_1} x + \mathsf{distY}^-_{t_1} + \mathsf{deltaY}_{t_1}$$
$$\iff t_0 + t_1 \leq (\underbrace{\mathsf{oslp}_{t_0} + \mathsf{oslp}_{t_1}}_{\mathsf{oslp}_{t_0+t_1}})x + \underbrace{\mathsf{distY}^-_{t_0} + \mathsf{distY}^-_{t_1}}_{\mathsf{distY}^-_{t_0+t_1}} + \underbrace{\mathsf{deltaY}_{t_0} + \mathsf{deltaY}_{t_1}}_{\mathsf{deltaY}_{t_0+t_1}}$$

⊣

inductive case $t = t_0 + t_1$.

inductive case $t = kt_0$:

By I.H. we have

$$\mathsf{oslp}_{t_0} \cdot x + \mathsf{distY}^-_{t_0} \leq t_0 \leq \mathsf{oslp}_{t_0} \cdot x + \mathsf{distY}^-_{t_0} + \mathsf{deltaY}_{t_0}$$

case $k \geq 0$:

Thus we have $k = |k|$ and can reason as follows:

$$\mathsf{oslp}_{t_0} \cdot x + \mathsf{distY}^-_{t_0} \leq \ t_0 \ \leq \mathsf{oslp}_{t_0} \cdot x + \mathsf{distY}^-_{t_0} + \mathsf{deltaY}_{t_0}$$
$$\iff \underbrace{k\mathsf{oslp}_{t_0}}_{\mathsf{oslp}_{kt_0}} \cdot x + \underbrace{k\mathsf{distY}^-_{t_0}}_{\mathsf{distY}^-_{kt_0}} \leq kt_0 \leq \underbrace{k\mathsf{oslp}_{t_0}}_{\mathsf{oslp}_{kt_0}} \cdot x + \underbrace{k\mathsf{distY}^-_{t_0}}_{\mathsf{distY}^-_{kt_0}} + \underbrace{k\mathsf{deltaY}_{t_0}}_{\mathsf{deltaY}_{kt_0}}$$

⊣

case $k \geq 0$.

case $k < 0$:

We can reason as follows:

$$\begin{aligned} & \mathsf{oslp}_{t_0} \cdot x + \mathsf{distY}^-_{t_0} \le t_0 \le \mathsf{oslp}_{t_0} \cdot x + \mathsf{distY}^-_{t_0} + \mathsf{deltaY}_{t_0} \\ \iff & \underbrace{k \cdot \mathsf{oslp}_{t_0}}_{\mathsf{oslp}_{kt_0}} \cdot x + k \cdot \mathsf{distY}^-_{t_0} \ge kt_0 \ge \underbrace{k \cdot \mathsf{oslp}_{t_0}}_{\mathsf{oslp}_{kt_0}} \cdot x + \underbrace{k \cdot \mathsf{distY}^-_{t_0} + k \cdot \mathsf{deltaY}_{t_0}}_{\mathsf{distY}^-_{kt_0}} \end{aligned}$$

Thus from $k < 0$ we know that $k + |k| = 0$, which means we can further reason as follows

$$\begin{aligned} \iff & \mathsf{oslp}_{kt_0} \cdot x + k \cdot \mathsf{distY}^-_{t_0} \ge kt_0 \\ \iff & \mathsf{oslp}_{kt_0} \cdot x + \underbrace{k \cdot \mathsf{distY}^-_{t_0} + k \cdot \mathsf{deltaY}_{t_0}}_{\mathsf{distY}^-_{kt_0}} + \underbrace{|k|\mathsf{deltaY}_{t_0}}_{\mathsf{deltaY}_{kt_0}} \ge kt_0 \end{aligned}$$

⊣

case $k < 0$.

inductive case $t = kt_0$.

inductive case $t = \lfloor t_0 \rfloor$:

By I.H. we have

$$\mathsf{oslp}_{t_0} \cdot x + \mathsf{distY}^-_{t_0} \le t_0 \le \mathsf{oslp}_{t_0} \cdot x + \mathsf{distY}^-_{t_0} + \mathsf{deltaY}_{t_0}$$

We can reason as follows

$$\begin{aligned} & \mathsf{oslp}_{t_0} \cdot x + \mathsf{distY}^-_{t_0} \le t_0 \\ \implies & \lfloor \mathsf{oslp}_{t_0} \cdot x + \mathsf{distY}^-_{t_0} \rfloor \le \lfloor t_0 \rfloor \\ \implies & \underbrace{\mathsf{oslp}_{t_0}}_{\mathsf{oslp}_{\lfloor t_0 \rfloor}} \cdot x + \underbrace{\mathsf{distY}^-_{t_0} - 1}_{\mathsf{distY}^-_{\lfloor t_0 \rfloor}} < \lfloor t_0 \rfloor \end{aligned}$$

and

$$\begin{aligned} & t_0 \le \mathsf{oslp}_{t_0} \cdot x + \mathsf{distY}^-_{t_0} + \mathsf{deltaY}_{t_0} \\ \implies & \lfloor t_0 \rfloor \le \mathsf{oslp}_{t_0} \cdot x + \mathsf{distY}^-_{t_0} + \mathsf{deltaY}_{t_0} \\ \implies & \lfloor t_0 \rfloor \le \underbrace{\mathsf{oslp}_{t_0}}_{\mathsf{oslp}_{\lfloor t_0 \rfloor}} \cdot x + \underbrace{\mathsf{distY}^-_{t_0} - 1}_{\mathsf{distY}^-_{\lfloor t_0 \rfloor}} + \underbrace{\mathsf{deltaY}_{t_0} + 1}_{\mathsf{deltaY}_{\lfloor t_0 \rfloor}} \end{aligned}$$

⊣

inductive case $t = \lfloor t_0 \rfloor$.

□

Lemma 12 (Grid Rounding)**.** *Let s, t be* Lira*-Terms, and $p \in \mathbb{Q}^{>0}$ then*

$$\begin{aligned} \lceil t \rceil^{s+p\mathbb{Z}} &= \max\{s + pz \mid z \in \mathbb{Z}, s + pz \le t\} \\ \lfloor t \rfloor^{s+p\mathbb{Z}} &= \min\{s + pz \mid z \in \mathbb{Z}, t \le s + pz\} \\ \lceil t + \varepsilon \rceil^{s+p\mathbb{Z}} &= \max\{s + pz \mid z \in \mathbb{Z}, s + pz < t\} \\ \lfloor t - \varepsilon \rfloor^{s+p\mathbb{Z}} &= \min\{s + pz \mid z \in \mathbb{Z}, t < s + pz\} \end{aligned}$$

Proof. By Quotient Remainder (Lem. 10) we get

$$\lceil t \rceil^{s+p\mathbb{Z}} \approx t + \underbrace{\mathsf{rem}_p(s-t)}_{\in[0,p)} \qquad \lfloor t \rfloor^{s+p\mathbb{Z}} \approx t - \underbrace{\mathsf{rem}_p(t-s)}_{\in[0,p)}$$

$$\approx s + p \cdot \underbrace{\mathsf{quot}_p(s-t)}_{\in\mathbb{Z}} \qquad \approx s - p \cdot \underbrace{\mathsf{quot}_p(t-s)}_{\in\mathbb{Z}}$$

As $\lceil t \rceil^{s+p\mathbb{Z}} \in [t, t+p)$ we know that $\lceil t \rceil^{s+p\mathbb{Z}}$ is the least value in $\{s+pz \mid z \in \mathbb{Z}, t \leq s+pz\}$. As $\lfloor t \rfloor^{s+p\mathbb{Z}} \in (t-p, t]$ we know that $\lfloor t \rfloor^{s+p\mathbb{Z}}$ is the greatest value in $\{s+pz \mid z \in \mathbb{Z}, s+pz \leq t\}$.

Further as

$$\lceil t+\varepsilon \rceil^{s+p\mathbb{Z}} = \lceil t+p \rceil^{s+p\mathbb{Z}} \qquad \lfloor t-\varepsilon \rfloor^{s+p\mathbb{Z}} = \lfloor t-p \rfloor^{s+p\mathbb{Z}}$$

from the results before know that $\lceil t+\varepsilon \rceil^{s+p\mathbb{Z}}$ is the least value in $\{s+pz \mid z \in \mathbb{Z}, t+p \leq s+pz\}$ and $\lfloor t-\varepsilon \rfloor^{s+p\mathbb{Z}}$ is the greatest value in $\{s+pz \mid z \in \mathbb{Z}, s+pz \leq t+p\}$. Thus the lemma holds. □

Lemma 3 (Grid Intersection; repeated)**.** $(s+p\mathbb{Z}) \sqcap (\!|t, t+k|\!) \supseteq (\{s+pz \mid z \in \mathbb{Z}\} \cap (\!|t, t+k|\!))$

Proof. **case $(\!|$ is $[$:**

By Grid Rounding (Lem. 12) we know that $\lceil t \rceil^{s+p\mathbb{Z}}$ is the least value in $\{s+pz \mid z \in \mathbb{Z}, t \leq s+pz\}$. Therefore the lemma obviously holds.

⊣

case $(\!|$ is $[$.

case $(\!|$ is $($:

By Grid Rounding (Lem. 12) we know that $\lceil t+\varepsilon \rceil^{s+p\mathbb{Z}}$ is the least value in $\{s+pz \mid z \in \mathbb{Z}, t < s+pz\}$. Therefore the lemma obviously holds.

⊣

case $(\!|$ is $($.

□

Lemma 13 (Limit Period)**.** *For any* Lira*-term t we have* $\mathsf{per}_{\mathsf{lim}_t} = \mathsf{per}_t$, $\mathsf{sslp}_{\mathsf{lim}_t} = \mathsf{sslp}_t$ *and* $\mathsf{oslp}_{\mathsf{lim}_t} = \mathsf{oslp}_t$.

Proof. We can apply induction on t and just unfold definitions. □

Lemma 14 (Breaks Period)**.** *For* $b + p\mathbb{Z} \in \mathsf{breaks}_t$ *we have* $\frac{\mathsf{per}_t}{p} \in \mathbb{Z}$.

Proof. We can apply induction on t and just unfold definitions. □

Lemma 4 (Piecewise Linearity; repeated)**.** *Let $\mathcal{I}$ be an $\mathbb{R}$-interpretation, $x \in \mathbf{V}$ and t a* Lira*-term such that* $\mathsf{breaks} \neq \emptyset$ *and* $b^- \in \mathsf{breaks}^\infty$*. Let* $b^+ = \min\{b \mid b \in \mathsf{breaks}^\infty, \mathcal{I} \models b > b^-\}$*, and* $\pm \in \{+, -\}$*. Then*

$$\mathcal{I} \models \forall x \in (b^-, b^+), y \in [b^-, b^+). (t[x] \approx \mathsf{lim}_t[x] \approx \mathsf{sslp} \cdot x + \mathsf{dseg}(y)) .$$

Proof. Consider the property P_1:

$$\forall x \in (b^-, b^+). t[x] \approx \mathsf{lim}_t[x] \approx \mathsf{sslp}_t(x - b^-) + \mathsf{lim}_t[b^-] \quad (P_1)$$

If we establish P_1, then for any $y \in (b^-, b^+)$ we can reason as follows

$$\begin{aligned} t[x] \approx \mathsf{lim}_t[x] &\approx \mathsf{sslp}_t x - \mathsf{sslp}_t b^- + \mathsf{lim}_t[b^-] \\ &\approx \mathsf{sslp}_t x \underbrace{-\mathsf{sslp}_t b^- + \mathsf{lim}_t[b^-] + \mathsf{sslp}_t y}_{\approx \mathsf{lim}_t[y] \text{by } P_1} - \mathsf{sslp}_t y \\ &\approx \mathsf{sslp}_t x + \underbrace{\mathsf{lim}_t[y] - \mathsf{sslp}_t y}_{\mathsf{dseg}_t(y)} \end{aligned}$$

This means that to proof Piecewise Linearity it is enough to show P_1. Thus we proceed by induction on t. We reason in an arbitrary $\mathbb{R}$-interpretation $\mathcal{I}$. If $\mathsf{breaks} = \emptyset$ then Piecewise Linearity trivially holds, hence let us only consider the cases where $\mathsf{breaks} \neq \emptyset$. Consider any $b^- \in \mathsf{breaks}_t^\infty$, $b^+ = \min\{b \mid b \in \mathsf{breaks}_t^\infty, \mathcal{I} \models b > b^-\}$, and any $x \in (b^-, b^+)$. Let us case split on t.

inductive case $t = t_0 + t_1$:

Then $\mathsf{breaks}_{t_0+t_1} = \mathsf{breaks}_{t_0} \cup \mathsf{breaks}_{t_1}$.

case $\mathsf{breaks}_{t_0} = \emptyset$ & $\mathsf{breaks}_{t_1} \neq \emptyset$:

First we observe that by Linear Term (Lem. 11), for any y we get

$$\begin{aligned} t_0[x] &\approx \mathsf{sslp}_{t_0} x + \mathsf{lim}_{t_0}[0] \\ &\approx \mathsf{sslp}_{t_0} x + \underbrace{\mathsf{lim}_{t_0}[0] + \mathsf{sslp}_{t_0} y}_{t_0[y] \approx \mathsf{lim}_{t_0}[y]} - \mathsf{sslp}_{t_0} y \\ &\approx \mathsf{sslp}_{t_0} x + \underbrace{\mathsf{lim}_{t_0}[y] - \mathsf{sslp}_{t_0} y}_{\mathsf{dseg}_{t_0}(y)} \qquad (P_2) \end{aligned}$$

As $\mathsf{breaks}_{t_0+t_1} \supseteq \mathsf{breaks}_{t_1}$, we can use I.H. to get

$$\begin{aligned} & t_1[x] \approx \mathsf{lim}_{t_1}[x] \approx \mathsf{sslp}_{t_1} x + \mathsf{dseg}_{t_1}(b^-) && \text{by I.H.} \\ \iff\ & t_0[x] + t_1[x] \approx t_0[x] + \mathsf{lim}_{t_1}[x] \approx t_0[x] + \mathsf{sslp}_{t_1} x + \mathsf{dseg}_{t_1}(b^-) \\ \iff\ & t_0[x] + t_1[x] \approx \mathsf{lim}_{t_0}[x] + \mathsf{lim}_{t_1}[x] \approx t_0[x] + \mathsf{sslp}_{t_1} x + \mathsf{dseg}_{t_1}(b^-) && \text{by Linear Term (Lem. 11)} \\ \iff\ & t_0[x] + t_1[x] \approx \mathsf{lim}_{t_0}[x] + \mathsf{lim}_{t_1}[x] \approx \mathsf{sslp}_{t_0} x + \mathsf{dseg}_{t_0}(b^-) + \mathsf{sslp}_{t_1} x + \mathsf{dseg}_{t_1}(b^-) && \text{by } P_2 \\ \iff\ & (t_0 + t_1)[x] \approx \mathsf{lim}_{t_0+t_1}[x] \approx \underbrace{(\mathsf{sslp}_{t_0} + \mathsf{sslp}_{t_1})}_{\mathsf{sslp}_{t_0+t_1}} \cdot x + \underbrace{\mathsf{dseg}_{t_0}(b^-) + \mathsf{dseg}_{t_1}(b^-)}_{\mathsf{dseg}_{t_0+t_1}(b^-)} \end{aligned}$$

⊣

case $\mathsf{breaks}_{t_0} = \emptyset$ & $\mathsf{breaks}_{t_1} \neq \emptyset$.

case $\mathsf{breaks}_{t_1} = \emptyset$ & $\mathsf{breaks}_{t_0} \neq \emptyset$:

This case can be shown in the same way as the case before.

⊣

case $\mathsf{breaks}_{t_1} = \emptyset$ & $\mathsf{breaks}_{t_0} \neq \emptyset$.

case $\mathsf{breaks}_{t_0} \neq \emptyset$ & $\mathsf{breaks}_{t_1} \neq \emptyset$**:**

As $\mathsf{breaks}_{t_0+t_1} \supseteq \mathsf{breaks}_{t_0}, \mathsf{breaks}_{t_1}$, we can use I.H. to get

$$\begin{aligned} & t_0[x] \approx \mathsf{lim}_{t_0}[x] \approx \mathsf{sslp}_{t_0} x + \mathsf{dseg}_{t_0}(b^-) \\ \& \quad & t_1[x] \approx \mathsf{lim}_{t_1}[x] \approx \mathsf{sslp}_{t_1} x + \mathsf{dseg}_{t_1}(b^-) \\ \Longrightarrow\ & t_0[x] + t_1[x] \approx \mathsf{lim}_{t_0}[x] + \mathsf{lim}_{t_1}[x] \approx \mathsf{sslp}_{t_0} x + \mathsf{dseg}_{t_0}(b^-) + \mathsf{sslp}_{t_1} x + \mathsf{dseg}_{t_1}(b^-) \\ \Longrightarrow\ & (t_0 + t_1)[x] \approx \mathsf{lim}_{t_0+t_1}[x] \approx \underbrace{(\mathsf{sslp}_{t_0} + \mathsf{sslp}_{t_1})}_{\mathsf{sslp}_{t_0+t_1}} x + \underbrace{\mathsf{dseg}_{t_0}(b^-) + \mathsf{dseg}_{t_1}(b^-)}_{\mathsf{dseg}_{t_0+t_1}(b^-)} \end{aligned}$$

⊣

case $\mathsf{breaks}_{t_0} \neq \emptyset$ & $\mathsf{breaks}_{t_1} \neq \emptyset$.

inductive case $t = t_0 + t_1$.

inductive case $t = kt_0$**:**

As $\mathsf{breaks}_{t_0} = \mathsf{breaks}_{kt_0}$ we can use I.H. to we get

$$t_0[x] \approx \mathsf{lim}_{t_0}[x] \approx \mathsf{sslp}_{t_0} x + \mathsf{dseg}_{t_0}(b^-) \Longrightarrow \underbrace{kt_0[x]}_{(kt_0)[x]} \approx \underbrace{k\mathsf{lim}_{t_0}[x]}_{\mathsf{lim}_{kt_0}[x]} \approx \underbrace{k\mathsf{sslp}_{t_0}}_{\mathsf{sslp}_{kt_0}} x + \underbrace{k\mathsf{dseg}_{t_0}(b^-)}_{\mathsf{dseg}_{kt_0}(b^-)}$$

⊣

inductive case $t = kt_0$.

inductive case $t = \lfloor t_0 \rfloor$**:**

As $\mathsf{sslp}_{\lfloor t_0 \rfloor} = 0$ we only need to establish $\lfloor t_0[x] \rfloor \approx \mathsf{lim}_{\lfloor t_0 \rfloor}[x] \approx \mathsf{lim}_{\lfloor t_0 \rfloor}[b^-]$ in order to show P_1 and thereby show Piecewise Linearity.

case $\mathsf{sslp}_{t_0} = 0$**:**

Then $\mathsf{breaks}_{t_0} = \mathsf{breaks}_{\lfloor t_0 \rfloor}$, thus we can use I.H. to get

$$\begin{aligned} t_0[x] \approx \mathsf{lim}_{t_0}[x] & \approx \mathsf{sslp}_{t_0} x + \mathsf{dseg}_{t_0}(b^-) \\ & \approx \underbrace{\mathsf{sslp}_{t_0}}_{0} x - \underbrace{\mathsf{sslp}_{t_0}}_{0} \cdot b^- + \mathsf{lim}_{t_0}[b^-] && \text{by Segment Line (Def. 4)} \\ \Longrightarrow \lfloor t_0[x] \rfloor \approx \lfloor \mathsf{lim}_{t_0}[x] \rfloor & \approx \lfloor \mathsf{lim}_{t_0}[b^-] \rfloor \\ \Longrightarrow \lfloor t_0[x] \rfloor \approx \mathsf{lim}_{\lfloor t_0 \rfloor}[x] & \approx \mathsf{lim}_{\lfloor t_0 \rfloor}[b^-] && \text{by Limit (Def. 3)} \end{aligned}$$

⊣

case $\mathsf{sslp}_{t_0} = 0$.

case $\mathsf{sslp}_{t_0} \neq 0$ & $\mathsf{breaks}_{t_0} = \emptyset$**:**

We have $\mathsf{breaks}^{\infty}_{\lfloor t_0 \rfloor} = \{\mathsf{zero}_{t_0}(0) + \mathsf{per}_{\lfloor t_0 \rfloor} z \mid z \in \mathbb{Z}\}$, with $\mathsf{per}_{\lfloor t_0 \rfloor} = \frac{1}{|\mathsf{oslp}_{t_0}|}$. Further by Linear Term (Lem. 11) we know that $\mathsf{zero}_{t_0}(0) \approx -\frac{t_0[0]}{\mathsf{sslp}_{t_0}}$. Therefore we know that for some $z \in \mathbb{Z}$

we have $b^- = \frac{-t_0[0]+z}{\mathsf{sslp}_{t_0}}$ and $b^+ = b^- + \frac{1}{|\mathsf{sslp}_{t_0}|}$. Thus by Linear Term (Lem. 11) that

$$\begin{aligned} t_0[x] &\approx \mathsf{sslp}_{t_0} x + t_0[0] \\ &\approx \mathsf{sslp}_{t_0} x + t_0[0] - \mathsf{sslp}_{t_0} b^- + \mathsf{sslp}_{t_0} \cdot \underbrace{\frac{-t_0[0]+z}{\mathsf{sslp}_{t_0}}}_{b^-} \\ &\approx \mathsf{sslp}_{t_0} \underbrace{(x - b^-)}_{\in(0,\frac{1}{|\mathsf{sslp}_{t_0}|})} + z \end{aligned} \qquad (P_3)$$

case $\mathsf{sslp}_{t_0} > 0$**:**

Then we can reason as follows

$$\begin{aligned} \mathsf{lim}_{\lfloor t_0 \rfloor}[b^-] &\approx \lfloor \mathsf{lim}_{t_0}[b^-] \rfloor && \text{by Limit (Def. 3)} \\ &\approx \lfloor \mathsf{sslp}_{t_0} \underbrace{\frac{-t_0[0]+z}{\mathsf{sslp}_{t_0}}}_{b^-} + t_0[0] \rfloor && \text{by Linear Term (Lem. 11)} \\ &\approx z \end{aligned}$$

$$\begin{aligned} \lfloor t_0[x] \rfloor &\approx \lfloor z + \underbrace{\mathsf{sslp}_{t_0}(x - b^-)}_{\in(0,1)} \rfloor && \text{by } P_3 \\ &\approx z \end{aligned}$$

$$\begin{aligned} \mathsf{lim}_{\lfloor x \rfloor}[t_0] &\approx \lfloor \mathsf{lim}_{t_0}[x] \rfloor && \text{by Limit (Def. 3)} \\ &\approx \lfloor z + \underbrace{\mathsf{sslp}_{t_0}(x - b^-)}_{\in(0,1)} \rfloor && \text{by } P_3 \\ &\approx z \end{aligned}$$

Thus we get $\mathsf{lim}_{\lfloor t_0 \rfloor}[b^-] \approx \lfloor t_0[x] \rfloor \approx \mathsf{lim}_{\lfloor x \rfloor}[t_0]$.

⊣

case $\mathsf{sslp}_{t_0} > 0$.

case $\mathsf{sslp}_{t_0} < 0$**:**

Then we can reason as follows

$$\begin{aligned} \mathsf{lim}_{\lfloor t_0 \rfloor}[b^-] &\approx \lceil \mathsf{lim}_{t_0}[b^-] \rceil - 1 && \text{by Limit (Def. 3)} \\ &\approx \lceil \mathsf{sslp}_{t_0} \underbrace{\frac{-t_0[0]+z}{\mathsf{sslp}_{t_0}}}_{b^-} + t_0[0] \rceil - 1 && \text{by Linear Term (Lem. 11)} \\ &\approx z - 1 \end{aligned}$$

$$\begin{aligned} \lfloor t_0[x] \rfloor &\approx \lfloor z + \underbrace{\mathsf{sslp}_{t_0}(x - b^-)}_{\in(-1,0)} \rfloor && \text{by } P_3 \\ &\approx z - 1 \end{aligned}$$

$$\begin{aligned} \mathsf{lim}_{\lfloor x \rfloor}[t_0] &\approx \lceil \mathsf{lim}_{t_0}[x] \rceil - 1 && \text{by Limit (Def. 3)} \\ &\approx \lceil z + \underbrace{\mathsf{sslp}_{t_0}(x - b^-)}_{\in(-1,0)} \rceil - 1 && \text{by } P_3 \\ &\approx z - 1 \end{aligned}$$

Thus we get $\mathsf{lim}_{\lfloor t_0 \rfloor}[b^-] \approx \lfloor t_0[x] \rfloor \approx \mathsf{lim}_{\lfloor x \rfloor}[t_0]$.

⊣

case $\mathsf{sslp}_{t_0} < 0$.

case $\mathsf{sslp}_{t_0} \neq 0$ & $\mathsf{breaks}_{t_0} = \emptyset$.

case $\mathsf{sslp}_{t_0} \neq 0$ & $\mathsf{breaks}_{t_0} \neq \emptyset$:

Notice that firstly $\mathsf{breaks}^\infty_{t_0} \subseteq \mathsf{breaks}^\infty_{\lfloor t_0 \rfloor}$. Let us firstly establish the property P_4:

lemma P_4:

$$\left(\begin{array}{rl} & b \in \mathsf{breaks}^\infty_{t_0} \\ \& & c = \min\{c \in \mathsf{breaks}^\infty_{t_0} \mid b < c\} \\ \& & z \in \mathbb{Z} \\ \& & \mathsf{zero}(b) + \frac{1}{\mathsf{sslp}_{t_0}} z \in [b, c) \\ \Longrightarrow & \mathsf{zero}(b) + \frac{1}{\mathsf{sslp}_{t_0}} z \in \mathsf{breaks}^\infty_{\lfloor t_0 \rfloor} \end{array}\right) \qquad (P_4)$$

P_4 says that for any two adjescent breaks $b, c \in \mathsf{breaks}_{t_0}$, any value d of the form $\mathsf{zero}(b) + \frac{1}{\mathsf{sslp}_{t_0}} z$ with $z \in \mathbb{Z}$, $d \in [b, c)$ is contained in $\mathsf{breaks}_{\lfloor t_0 \rfloor}$. Intuitively the values d are all the values where the line segment of t_0 in the interval $[b, c)$ takes an integer value.

Consider any $b \in \mathsf{breaks}^\infty_{t_0}$, and $c = \min\{c \in \mathsf{breaks}^\infty_{t_0} \mid b < c\}$. There must be some $t_0 + p^{\min}_{t_0}\mathbb{Z} \in \mathsf{breaks}_{t_0}$. Thus there is some $t_0 + p^{\min}_{t_0} z \in \mathsf{breaks}^\infty_{t_0}$ such that $t_0 + p^{\min}_{t_0} z - b \leq p^{\min}_{t_0}$ which means that $c - b \leq p^{\min}_{t_0}$ as c is minimal, which further means that $[b, c) \subseteq [b, b + p^{\min}_{t_0})$. We can reason as follows

$$\begin{array}{rlll} & b & \in \mathsf{breaks}^\infty_{t_0} & \\ \Longrightarrow & b & \approx b'_0 + pz & \text{for some } b'_0 + p\mathbb{Z} \in \mathsf{breaks}_{t_0} \\ \Longleftrightarrow & b & \approx b'_0 \begin{array}{l} + \mathsf{per}_{\lfloor t_0 \rfloor} \overbrace{\mathsf{quot}_{\mathsf{per}_{\lfloor t_0 \rfloor}}(pz)}^{\in \mathbb{Z}} \\ + \underbrace{\mathsf{rem}_{\mathsf{per}_{\lfloor t_0 \rfloor}}(pz)}_{\in [0, \mathsf{per}_{\lfloor t_0 \rfloor})} \end{array} & \text{by Quotient Remainder (Lem. 10)} \end{array}$$

$$\begin{array}{rll} \mathsf{rem}_{\mathsf{per}_{\lfloor t_0 \rfloor}}(pz) & \approx pz - \mathsf{per}_{\lfloor t_0 \rfloor} \lfloor \frac{pz}{\mathsf{per}_{\lfloor t_0 \rfloor}} \rfloor & \\ & \approx pz - p \underbrace{\frac{\mathsf{per}_{\lfloor t_0 \rfloor}}{p}}_{\in \mathbb{Z}} \lfloor \frac{pz}{\mathsf{per}_{\lfloor t_0 \rfloor}} \rfloor & \text{by Breaks Period (Lem. 14)} \\ & \approx pz - p \lfloor \frac{\mathsf{per}_{\lfloor t_0 \rfloor}}{p} \lfloor \frac{pz}{\mathsf{per}_{\lfloor t_0 \rfloor}} \rfloor \rfloor & \\ & \approx p \lfloor z - \lfloor \frac{\mathsf{per}_{\lfloor t_0 \rfloor}}{p} \lfloor \frac{pz}{\mathsf{per}_{\lfloor t_0 \rfloor}} \rfloor \rfloor \rfloor & \end{array}$$

Thus we know that $b \approx b'_0 + pz_p + \mathsf{per}_{\lfloor t_0 \rfloor} z_{\mathsf{per}_{\lfloor t_0 \rfloor}}$ for some $z_p, z_{\mathsf{per}_{\lfloor t_0 \rfloor}} \in \mathbb{Z}$ with $pz_p \in [0, \mathsf{per}_{\lfloor t_0 \rfloor})$. We define $b_0 = b'_0 + pz_p$, and thus know by Grid Intersection (Lem. 3) that $b_0 \in (b'_0 + p\mathbb{Z}) \sqcap [b'_0, b'_0 + \mathsf{per}_{\lfloor t_0 \rfloor})$.

Now consider any $z' \in \mathbb{Z}$ such that $\mathsf{zero}(b) + \frac{z'}{\mathsf{sslp}_{t_0}} \in [b, c) \subseteq [b, b + p^{\min}_{t_0})$. We can reason as follows:

$$
\begin{aligned}
&\mathsf{zero}(\overbrace{b_0 + \mathsf{per}_{\lfloor t_0 \rfloor} z_{\mathsf{per}_{\lfloor t_0 \rfloor}}}^{b}) + \tfrac{z'}{\mathsf{sslp}_{t_0}} \\
\approx & b_0 + \mathsf{per}_{\lfloor t_0 \rfloor} z_{\mathsf{per}_{\lfloor t_0 \rfloor}} - \tfrac{\mathsf{lim}_{t_0}[b_0 + \mathsf{per}_{\lfloor t_0 \rfloor} z_{\mathsf{per}_{\lfloor t_0 \rfloor}}]}{\mathsf{sslp}_{t_0}} + \tfrac{z'}{\mathsf{sslp}_{t_0}} \\
\approx & b_0 + \mathsf{per}_{\lfloor t_0 \rfloor} z_{\mathsf{per}_{\lfloor t_0 \rfloor}} - \tfrac{\mathsf{lim}_{t_0}[b_0] + \mathsf{oslp}_{t_0} \mathsf{per}_{\lfloor t_0 \rfloor} z_{\mathsf{per}_{\lfloor t_0 \rfloor}}}{\mathsf{sslp}_{t_0}} + \tfrac{z'}{\mathsf{sslp}_{t_0}} \quad \text{by Limit Period (Lem. 13)} \\
\approx & b_0 + \mathsf{per}_{\lfloor t_0 \rfloor} z_{\mathsf{per}_{\lfloor t_0 \rfloor}} - \tfrac{\mathsf{lim}_{t_0}[b_0]}{\mathsf{sslp}_{t_0}} + \tfrac{z' - \mathsf{oslp}_{t_0} \mathsf{per}_{\lfloor t_0 \rfloor} z_{\mathsf{per}_{\lfloor t_0 \rfloor}}}{\mathsf{sslp}_{t_0}} \\
\approx & \mathsf{zero}(b_0) + \mathsf{per}_{\lfloor t_0 \rfloor} z_{\mathsf{per}_{\lfloor t_0 \rfloor}} + \tfrac{z' - \mathsf{oslp}_{t_0} \mathsf{per}_{\lfloor t_0 \rfloor} z_{\mathsf{per}_{\lfloor t_0 \rfloor}}}{\mathsf{sslp}_{t_0}} \qquad (P_5)
\end{aligned}
$$

We define $z'' = z' - \mathsf{oslp}_{t_0} \mathsf{per}_{\lfloor t_0 \rfloor} z_{\mathsf{per}_{\lfloor t_0 \rfloor}}$. By Slope and Period (Def. 1) we know that $\mathsf{oslp}_{t_0} \mathsf{per}_{\lfloor t_0 \rfloor} \in \mathbb{Z}$, thus we know that $z'' \in \mathbb{Z}$. Further as $\mathsf{zero}(b) + \frac{z'}{\mathsf{sslp}_{t_0}} \in [b, b + p_{t_0}^{\min})$ we can continue reasoning as follows

$$
\begin{aligned}
& 0 \le \mathsf{zero}(b) + \frac{z'}{\mathsf{sslp}_{t_0}} - (\overbrace{b_0 + \mathsf{per}_{\lfloor t_0 \rfloor} z_{\mathsf{per}_{\lfloor t_0 \rfloor}}}^{b}) < p_{t_0}^{\min} \\
\iff & 0 \le \mathsf{zero}(b_0) + \cancel{\mathsf{per}_{\lfloor t_0 \rfloor} z_{\mathsf{per}_{\lfloor t_0 \rfloor}}} + \tfrac{z''}{\mathsf{sslp}_{t_0}} - b_0 - \cancel{\mathsf{per}_{\lfloor t_0 \rfloor} z_{\mathsf{per}_{\lfloor t_0 \rfloor}}} < p_{t_0}^{\min} \qquad \text{by } P_5 \\
\iff & \mathsf{zero}(b_0) + \tfrac{z''}{\mathsf{sslp}_{t_0}} \in [b_0, b_0 + p_{t_0}^{\min}) \\
\implies & \mathsf{zero}(b_0) + \tfrac{z''}{\mathsf{sslp}_{t_0}} \in \mathsf{breaks}_{\lfloor t_0 \rfloor}^{\infty} \\
\implies & \mathsf{zero}(b_0) + \tfrac{z''}{\mathsf{sslp}_{t_0}} + \mathsf{per}_{\lfloor t_0 \rfloor} \in \mathsf{breaks}_{\lfloor t_0 \rfloor}^{\infty} \\
\implies & \mathsf{zero}(b) + \tfrac{z'}{\mathsf{sslp}_{t_0}} \in \mathsf{breaks}_{\lfloor t_0 \rfloor}^{\infty} \qquad \text{by } P_5
\end{aligned}
$$

⊣

lemma P_4.

Then let a^- be the greatest value in $\mathsf{breaks}_{t_0}^{\infty}$ such that $a^- < b^-$, and a^+ be the least value in $\mathsf{breaks}_{t_0}^{\infty}$ such that $a- < a^+$.

We define $b = \mathsf{zero}(a^-) + \frac{z}{\mathsf{sslp}_{t_0}}$ to be the greatest such b such that $b \le a^-$.

By P_4 we know that either $a^+ \le b + \frac{1}{|\mathsf{sslp}_{t_0}|}$, or $b + \frac{1}{|\mathsf{sslp}_{t_0}|} \in \mathsf{breaks}_{\lfloor t_0 \rfloor}^{\infty}$. In either case there is some $b' \in \mathsf{breaks}_{\lfloor t_0 \rfloor}^{\infty}$ such that $0 < b' - b \le \frac{1}{|\mathsf{sslp}_{t_0}|}$, thus $0 < b^+ - b \le \frac{1}{|\mathsf{sslp}_{t_0}|}$ as b^+ is minimal, which means that $x \in [b, b + \frac{1}{|\mathsf{sslp}_{t_0}|})$.

Further by the minimality of b we know that

$$\begin{aligned}
& b \leq a^- < b + \tfrac{1}{|\mathsf{sslp}_{t_0}|} \\
\Longleftrightarrow & 0 \leq a^- - b < \tfrac{1}{|\mathsf{sslp}_{t_0}|} \\
\Longleftrightarrow & 0 \leq a^- - \mathsf{zero}(a^-) - \tfrac{z}{\mathsf{sslp}_{t_0}} < \tfrac{1}{|\mathsf{sslp}_{t_0}|} \\
\Longleftrightarrow & 0 \leq \cancel{a^- - a^-} + \tfrac{\mathsf{lim}_{t_0} a^-}{\mathsf{sslp}_{t_0}} - \tfrac{z}{\mathsf{sslp}_{t_0}} < \tfrac{1}{|\mathsf{sslp}_{t_0}|} && (P_6)
\end{aligned}$$

As $\mathsf{breaks}^{\infty}_{\lfloor t_0 \rfloor} \supseteq \mathsf{breaks}^{\infty}_{t_0}$ we can use I.H.to get

$$\begin{aligned}
t_0[x] & \approx \mathsf{sslp}_{t_0} x + \mathsf{dseg}_{t_0}(a^-) \\
& \approx \mathsf{sslp}_{t_0}(x - \mathsf{zero}_{t_0}(a^-)) \\
& \approx \mathsf{sslp}_{t_0}(x - (b - \frac{z}{\mathsf{sslp}_{t_0}})) && \text{by definition of } b \\
& \approx \mathsf{sslp}_{t_0}(\underbrace{x - b}_{\in(0, \frac{1}{|\mathsf{sslp}_{t_0}|})}) + z && (P_7)
\end{aligned}$$

case $\mathsf{sslp}_{t_0} > 0$**:**

$$\begin{aligned}
& 0 \leq \tfrac{\mathsf{lim}_{t_0}[a^-]}{\mathsf{sslp}_{t_0}} - \tfrac{z}{\mathsf{sslp}_{t_0}} < \tfrac{1}{|\mathsf{sslp}_{t_0}|} && \text{by } P_6 \\
\Longleftrightarrow \quad & 0 \leq \mathsf{lim}_{t_0}[a^-] - z < 1 \\
\Longrightarrow \quad & 0 \approx \lfloor \mathsf{lim}_{t_0}[a^-] - z \rfloor \\
\Longleftrightarrow \quad & z \approx \lfloor \mathsf{lim}_{t_0}[a^-] \rfloor \\
& \approx \mathsf{lim}_{\lfloor t_0 \rfloor}[a^-] && \text{by Limit (Def. 3)}
\end{aligned}$$

$$\begin{aligned}
\lfloor t_0[x] \rfloor & \approx \lfloor z + \underbrace{\mathsf{sslp}_{t_0}(x - b)}_{\in(0,1)} \rfloor && \text{by } P_7 \\
& \approx z
\end{aligned}$$

$$\begin{aligned}
\mathsf{lim}_{\lfloor t_0 \rfloor}[x] & \approx \lfloor \mathsf{lim}_{t_0}[x] \rfloor && \text{by Limit (Def. 3)} \\
& \approx \lfloor t_0[x] \rfloor && \text{by I.H.} \\
& \approx \lfloor z + \underbrace{\mathsf{sslp}_{t_0}(x - b)}_{\in(0,1)} \rfloor && \text{by } P_7 \\
& \approx z
\end{aligned}$$

Thus we get $\mathsf{lim}_{\lfloor t_0 \rfloor}[b^-] \approx \lfloor t_0[x] \rfloor \approx \mathsf{lim}_{\lfloor x \rfloor}[t_0]$.

⊣

case $\mathsf{sslp}_{t_0} > 0$.

case $\mathsf{sslp}_{t_0} < 0$**:**

$$
\begin{aligned}
& 0 \leq \frac{\mathsf{lim}_{t_0}[a^-]}{\mathsf{sslp}_{t_0}} - \frac{z}{\mathsf{sslp}_{t_0}} < \frac{1}{|\mathsf{sslp}_{t_0}|} && \text{by } P_6 \\
\iff\quad & 0 \geq \mathsf{lim}_{t_0}[a^-] - z > -1 \\
\implies\quad & 0 \approx \lceil \mathsf{lim}_{t_0}[a^-] - z \rceil \\
\iff\quad & z - 1 \approx \lceil \mathsf{lim}_{t_0}[a^-] \rceil - 1 \\
& \approx \mathsf{lim}_{\lfloor t_0 \rfloor}[a^-] && \text{by Limit (Def. 3)}
\end{aligned}
$$

$$
\begin{aligned}
\lfloor t_0[x] \rfloor & \approx \lfloor z + \underbrace{\mathsf{sslp}_{t_0}(x - b)}_{\in(-1,0)} \rfloor && \text{by } P_7 \\
& \approx z - 1
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{lim}_{\lfloor t_0 \rfloor}[x] & \approx \lceil \mathsf{lim}_{t_0}[x] \rceil - 1 && \text{by Limit (Def. 3)} \\
& \approx \lceil t_0[x] \rceil - 1 && \text{by I.H.} \\
& \approx \lceil z + \underbrace{\mathsf{sslp}_{t_0}(x - b)}_{\in(-1,0)} \rceil - 1 && \text{by } P_7 \\
& \approx z - 1
\end{aligned}
$$

Thus we get $\mathsf{lim}_{\lfloor t_0 \rfloor}[b^-] \approx \lfloor t_0[x] \rfloor \approx \mathsf{lim}_{\lfloor x \rfloor}[t_0]$.

⊣

case $\mathsf{sslp}_{t_0} < 0$.

case $\mathsf{sslp}_{t_0} \neq 0$ & $\mathsf{breaks}_{t_0} \neq \emptyset$.

inductive case $t = \lfloor t_0 \rfloor$.

□

Lemma 5 (Periodic Literals; repeated)**.** *If* $L = t \diamond 0$ *is a periodic* LIRA*-literal (*$\mathsf{oslp}_t = 0$*), then*

$$\mathbb{R} \models \forall y.\big(L[x] \leftrightarrow L[x + \mathsf{per}_t \lfloor y \rfloor]\big)$$

Proof. Follows straight from Periodic Shift (Lem. 1) if $\mathsf{breaks}_t \neq \emptyset$ and from Linear Term (Lem. 11) otherwise. □

Lemma 15 (Periodic Literal Induction)**.** *If* $L = t \diamond 0$ *is a periodic* LIRA*-literal (*$\mathsf{oslp}_t = 0$*), then*

$$\mathbb{R} \models \exists y.\big(\forall x \in [y, y + \mathsf{per}_t).L[x]\big) \to \forall x.L[x]$$

Proof. We reason in some arbitrary $\mathbb{R}$-interpreation $\mathcal{I}$. Assume that $\forall x.(y \leq x < y + \mathsf{per}_t \to t[x] \diamond 0)$, and let x be arbitrary. There is some z such that $x + \mathsf{per}_t \lfloor z \rfloor \in [y, y + \mathsf{per}_t)$. By Periodic Literals (Lem. 5) we have $L[x + \mathsf{per}_t \lfloor z \rfloor] \leftrightarrow L[x]$, thus we know that $L[x]$ holds. □

Lemma 6 (Limit Value; repeated)**.** *If* $L = t \diamond 0$ *is an aperiodic* LIRA*-literal (*$\mathsf{oslp}_t \neq 0$*), then the values outside of the core interval of* t *satisfy the following:*

$$\mathbb{R} \models \forall x < \mathsf{distX}_t^-.(L[x] \leftrightarrow \mathsf{lim}_L^{-\infty}) \qquad \mathbb{R} \models \forall x > \mathsf{distX}_t^+.(L[x] \leftrightarrow \mathsf{lim}_L^{+\infty})$$

where

$$\mathsf{lim}_{t \approx 0}^{\pm\infty} = \bot \qquad \mathsf{lim}_{t \not\approx 0}^{\pm\infty} = \top \qquad \mathsf{lim}_{t \gtrsim 0}^{\pm\infty} = \pm\mathsf{oslp} > 0$$

Proof. We reason in an arbitrary $\mathbb{R}$-interpretation $\mathcal{I}$.
case $\mathsf{sgn}(\mathsf{oslp}) = +$**:**

$$x < \underbrace{-\frac{\mathsf{distY}^+}{\mathsf{oslp}}}_{\mathsf{distX}^-} \iff \mathsf{oslp} \cdot x + \mathsf{distY}^+ < 0$$

and

$$x > \underbrace{-\frac{\mathsf{distY}^+}{\mathsf{oslp}} + \frac{\mathsf{deltaY}}{|\mathsf{oslp}|}}_{\mathsf{distX}^+} \iff \mathsf{oslp} \cdot x + \underbrace{\mathsf{distY}^+ - \mathsf{deltaY}}_{\mathsf{distY}^-} > 0$$

Thus by Linear Bounds (Lem. 2) we have (P_1) $x > \mathsf{distX}^+ \implies t > 0$, and $x < \mathsf{distY}^- \implies t < 0$.

case $\mathsf{sgn}(\mathsf{oslp}) = +$.

case $\mathsf{sgn}(\mathsf{oslp}) = -$:

$$x < \underbrace{-\frac{\mathsf{distY}^-}{\mathsf{oslp}}}_{\mathsf{distX}^-} \iff \mathsf{oslp} \cdot x + \mathsf{distY}^- > 0$$

and

$$x > \underbrace{-\frac{\mathsf{distY}^-}{\mathsf{oslp}} + \frac{\mathsf{deltaY}}{|\mathsf{oslp}|}}_{\mathsf{distX}^+} \iff \mathsf{oslp} \cdot x + \underbrace{\mathsf{distY}^- + \mathsf{deltaY}}_{\mathsf{distY}^+} < 0$$

Thus by Linear Bounds (Lem. 2) we have (P_2) $x > \mathsf{distX}^+ \implies t < 0$, and $x < \mathsf{distY}^- \implies t > 0$.

case $\mathsf{sgn}(\mathsf{oslp}) = -$.

We case split on $\diamond$.

case $\diamond$ is $\approx$:

Then $\mathsf{lim}^{\pm\infty}_{t\diamond 0} = \bot$. Thus the lemma follows straight from P_1 and P_2.

⊣

case $\diamond$ is $\approx$.

case $\diamond$ is $\not\approx$:

Then $\mathsf{lim}^{\pm\infty}_{t\diamond 0} = \top$. Thus the lemma follows straight from P_1 and P_2.

⊣

case $\diamond$ is $\not\approx$.

case $\diamond$ is $\gtrsim \in \{>, \geq\}$:

case $\mathsf{oslp} > 0$:

Then $\mathsf{lim}^{+\infty} = +\mathsf{oslp} > 0 = \top$, and $\mathsf{lim}^{-\infty} = -\mathsf{oslp} > 0 = \bot$, thus the lemma follows from P_1.

⊣

case $\mathsf{oslp} > 0$.

case $\mathsf{oslp} < 0$:

Then $\mathsf{lim}^{+\infty} = +\mathsf{oslp} > 0 = \bot$, and $\mathsf{lim}^{-\infty} = -\mathsf{oslp} > 0 = \top$, thus the lemma follows from P_2.

⊣

case $\mathsf{oslp} < 0$.

case $\diamond$ **is** $\gtrsim \in \{>, \geq\}$.

□

Lemma 16 (Zeros)**.** *Let s, t be* Lira*-terms, and $\diamond \in \{\approx, \not\approx, \geq, >\}$, then*

1. $\mathsf{sslp}_t > 0$ *then* $\mathbb{R} \models \mathsf{sslp}_t \cdot s + \mathsf{lim}_t[b] \gtrsim 0 \iff s \diamond \mathsf{zero}_t(b)$
2. $\mathsf{sslp}_t < 0$ *then* $\mathbb{R} \models \mathsf{sslp}_t \cdot s + \mathsf{lim}_t[b] \gtrsim 0 \iff \mathsf{zero}_t(b) \diamond s$

Proof. In both cases we can expand defintions to get

1. $\mathsf{sslp}_t \cdot s + \mathsf{lim}_t[b] \diamond 0 \iff s \diamond -\frac{\mathsf{lim}_t[b]}{\mathsf{sslp}_t} \iff s \diamond \mathsf{zero}_t(b)$
2. $\mathsf{sslp}_t \cdot s + \mathsf{lim}_t[b] \diamond 0 \iff -\frac{\mathsf{lim}_t[b]}{\mathsf{sslp}_t} \diamond s \iff \mathsf{zero}_t(b) \diamond s$

□

Lemma 17 (Lower Bound Existence)**.** *Let L be a* Lira*-literal, $\mathcal{I}$ be some $\mathbb{R}$-interpretation, such that $\mathcal{I} \models L[y]$. Then there is some $v \in \mathsf{elim}^x(L)$ such that $\mathcal{I} \models \mathsf{sInter}(v, y, L)$.*

Proof. Let $L = t \diamond 0$. We reason in the interpretation $\mathcal{I}$, and case split on L along the definition of elim.

case $\mathsf{breaks} = \emptyset$**:**

Then by Linear Term (Lem. 11), we know that $t \approx \mathsf{sslp} \cdot x + \mathsf{lim}_t[0]$.

case $\mathsf{sslp} = 0$**:**

Then by Linear Term (Lem. 11) we know that $t[y] \diamond 0 \iff t[x'] \diamond 0$ for any x'. Thus L holds for any $x' \in (-\infty, y]$ which obviously means that $[-\infty, y]$ is a solution interval, and $-\infty \in \mathsf{elim}(L)$.

⊣

case $\mathsf{sslp} = 0$.

case $\diamond$ **is** $\approx$**:**

Then by Zeros (Lem. 16) and Linear Term (Lem. 11), from $t[y] \approx 0$, we know that $y \approx \mathsf{zero}(0)$, thus $[\mathsf{zero}(0), y]$ is a solution interval, and $\mathsf{zero}(0) \in \mathsf{elim}(L)$.

⊣

case $\diamond$ **is** $\approx$.

case $\diamond$ **is** $\gtrsim$**:**

As $\mathsf{sslp} \neq 0$, by Zeros (Lem. 16) we know that $\mathsf{zero}(0)$ is the only zero of $t[x]$.

case $\mathsf{sslp} > 0$**:**

This means since t is a linear term (Linear Term (Lem. 11)), and $t[y] \gtrsim 0$ by Zeros (Lem. 16) we know that $y \gtrsim \mathsf{zero}(0)$.

Let x' be any value such that $\mathsf{zero}(0) \lesssim x' \leq y$. For any such value by Zeros (Lem. 16) we know that $t[x'] \gtrsim 0$. Thus if $\gtrsim$ is $>$ then $[\mathsf{zero}(0) + \varepsilon, y]$ is a solution interval, and if $\gtrsim$ is $\geq$, then $[\mathsf{zero}(0), y]$ is a solution interval, and the respective lower bound is in $\mathsf{elim}(L)$.

⊣

case $\mathsf{sslp} > 0$.

case $\mathsf{sslp} < 0$**:**

This means since t is a linear term (Linear Term (Lem. 11)), and $t[y] \gtrsim 0$ by Zeros (Lem. 16) that $y \lesssim \mathsf{zero}(0)$. Let x' be any value such that $x' \leq y$. By transitivity we ahve that $x' \lesssim \mathsf{zero}(0)$, thus by Zeros (Lem. 16) we hace $t[x'] \gtrsim 0$. Thus $[-\infty, y]$ is a solution interval, and $-\infty \in \mathsf{elim}(L)$.

⊣

case $\mathsf{sslp} < 0$.

case $\diamond$ **is** $\gtrsim$.

case $\diamond$ **is** $\not\approx$**:**

Then we know that $t[y] > 0$ or $t[y] < 0$. In both both cases we can reason as in the inequalitiy cases before to get that either $[\mathsf{zero}(0) + \varepsilon, y]$ is a solution interval, or $[-\infty, y]$ is a solution interval, with either of the two lower bounds being in $\mathsf{elim}(L)$.

⊣

case $\diamond$ **is** $\not\approx$.

case $\mathsf{breaks} = \emptyset$.

case $\mathsf{breaks} \neq \emptyset$**:**

case $\mathsf{oslp} = 0$**:**

We case split on the value y

case $\exists b^* \in \mathsf{breaks}^\infty . y \approx b^*$**:**

Then there is some value $b + p\mathbb{Z} \in \mathsf{breaks}$ such that $y \approx b + pz$ for some $z \in \mathbb{Z}$, thus $[b + p\mathbb{Z}, y]$ is a solution interval.

⊣

case $\exists b^* \in \mathsf{breaks}^\infty . y \approx b^*$.

case $\forall b^* \in \mathsf{breaks}^\infty . y \not\approx b^*$**:**

Let $b^- \in \mathsf{breaks}$, and $y' = \min\{y' \mid y' \approx y + z, b^- < y', z \in \mathbb{Z}\}$. By piecewise linearity we know that there is some b^+ such that s is a linear segment in the interval (b^-, b^+). As L is periodic by Periodic Literals (Lem. 5) we know that from $L[y]$ it follows that $L[y']$. Further again because L is periodic if we show that $[p, y']$ is a truth interval, we also know that $[p + p\mathbb{Z}, y]$ is a truth interval.

If $\mathsf{sslp} \neq 0$ the zero of this segment is $\mathsf{zero}(b^-)$. We case split:

case $\mathsf{sslp} = 0$**:**

From $t[y'] \diamond 0$, Piecewise Linearity (Lem. 4), we know that $t[x']$ holds for any $x' \in (b^-, y']$, thus $[b^- + \varepsilon, y']$ is a solution interval, with $b^- + \varepsilon \in \mathsf{elim}(L)$.

⊣

case $\mathsf{sslp} = 0$.

case $\mathsf{sslp} < 0$ & $\diamond$ **is** $\gtrsim \in \{>, \geq\}$**:**

In this case by Piecewise Linearity (Lem. 4), and Zeros (Lem. 16) we have $y' \lesssim \mathsf{zero}(b^-)$. Let $x' \in (b^-, y']$, then by transitivity we know that $x' \lesssim \mathsf{zero}(b^-)$, thus by Piecewise Linearity (Lem. 4), and Zeros (Lem. 16) we have $t[x'] \gtrsim 0$. Therefore we know that $[b^- + \varepsilon, y']$ is a solution interval with $b^- + \varepsilon \in \mathsf{elim}(L)$.

$\dashv$

case $\mathsf{sslp} < 0$ & $\diamond$ **is** $\gtrsim \in \{>, \geq\}$.

case $\mathsf{sslp} > 0$ & $\diamond$ **is** $\gtrsim \in \{>, \geq\}$ & $\mathsf{zero}(b^-) \notin (b^-, b^+)$**:**

By Piecewise Linearity (Lem. 4), and Zeros (Lem. 16) we know that $b^+ > y' \gtrsim \mathsf{zero}(b^-)$. As $\mathsf{zero}(b^-) \notin (b^-, b^+)$ we know that $b^+ > y' > b^- \geq \mathsf{zero}(b^-)$. Consider any $x' \in (b^-, y']$. By Piecewise Linearity (Lem. 4), and Zeros (Lem. 16) we know that $t[x'] > 0$, thus $[b^- + \varepsilon, y']$ is a solution interval with $b^- + \varepsilon \in \mathsf{elim}(L)$.

$\dashv$

case $\mathsf{sslp} > 0$ & $\diamond$ **is** $\gtrsim \in \{>, \geq\}$ & $\mathsf{zero}(b^-) \notin (b^-, b^+)$.

case $\mathsf{sslp} > 0$ & $\diamond$ **is** $\gtrsim \in \{>, \geq\}$ & $\mathsf{zero}(b) \in (b^-, b^+)$**:**

Consider some x' such that $\mathsf{zero}(b^-) \lesssim x' \leq y'$. Then by Piecewise Linearity (Lem. 4), and Zeros (Lem. 16) we know that $t[x'] \gtrsim 0$. Therefore if $\gtrsim$ is $>$ then $[\mathsf{zero}(b^-) + \varepsilon, y']$ is a solution interval, and if $\gtrsim$ is $\geq$, then $[\mathsf{zero}(b^-), y']$ is a solution interval with $\mathsf{zero}(b^-) \in \mathsf{elim} b^-$.

$\dashv$

case $\mathsf{sslp} > 0$ & $\diamond$ **is** $\gtrsim \in \{>, \geq\}$ & $\mathsf{zero}(b) \in (b^-, b^+)$.

case $\mathsf{sslp} \neq 0$ & $\diamond$ **is** $\not\approx$**:**

If $t[y'] \not\approx 0$ there are two cases, either $t[y'] > 0$ or $-t[y'] > 0$. In either case we can reason as in the cases where $\diamond$ is $>$, to get that either $[b^- + \varepsilon, y']$, or $[\mathsf{zero}(b^-) + \varepsilon, y']$ is a solution interval with $\mathsf{zero}(b^-) \in \mathsf{elim}(L)$.

$\dashv$

case $\mathsf{sslp} \neq 0$ & $\diamond$ **is** $\not\approx$.

case $\mathsf{sslp} \neq 0$ & $\diamond$ **is** $\approx$**:**

Then by Zeros (Lem. 16), and Piecewise Linearity (Lem. 4) it must be the case that $y' = \mathsf{zero}(b^-)$, hence $[\mathsf{zero}(b^-), y']$ is a solution interval, with $\mathsf{zero}(b^-) \in \mathsf{elim}(L)$

$\dashv$

case $\mathsf{sslp} \neq 0$ & $\diamond$ **is** $\approx$.

case $\forall b^* \in \mathsf{breaks}^\infty . y \not\approx b^*$.

case $\mathsf{oslp} = 0$.

case $\mathsf{oslp} \neq 0$**:**

We case split on y.

case $y \approx \mathsf{distX}^\pm$**:**

Then $\mathsf{distX}^\pm \in \mathsf{elim}(L)$ and $[\mathsf{distX}^\pm, y]$ is a solution interval.

$\dashv$

case $y \approx \mathsf{distX}^\pm$.

case $y \in (-\infty, \mathsf{distX}^-)$**:**

Then as $L[y]$ by Limit Value (Lem. 6) it must be the case that $\mathsf{lim}_L^{-\infty} = \top$, thus $-\infty \in \mathsf{elim}(L)$, and $[-\infty, y]$ is a solution interval.

⊣

case $y \in (-\infty, \mathsf{distX}^-)$.

case $y \in (\mathsf{distX}^+, \infty)$**:**

Then as $L[y]$ by Limit Value (Lem. 6) it must be the case that $\mathsf{lim}_L^{+\infty} = \top$, thus $\mathsf{distX}^+ + \varepsilon \in \mathsf{elim}(L)$, and $[\mathsf{distX}^+ + \varepsilon, y]$ is a solution interval.

⊣

case $y \in (\mathsf{distX}^+, \infty)$.

case $y \in (\mathsf{distX}^-, \mathsf{distX}^+)$ & $y \in \mathsf{breaks}^\infty$**:**

Then there is some value $b \in \mathsf{ebreak}$, such that $y \approx b$, thus $[b, y]$ is a solution interval.

⊣

case $y \in (\mathsf{distX}^-, \mathsf{distX}^+)$ & $y \in \mathsf{breaks}^\infty$.

case $y \in (\mathsf{distX}^-, \mathsf{distX}^+)$ & $y \notin \mathsf{breaks}^\infty$**:**

Then by Piecewise Linearity (Lem. 4) there $y \in (b^-, b^+)$, such that t is a linear segment in between these breaks b^-, b^+.
Notice the following:

$$
\begin{aligned}
& \{\ \mathsf{zero}(b) \mid b \in \mathsf{breaks}^\infty \ \} \\
= & \{\ b - \tfrac{\mathsf{lim}_t[b]}{\mathsf{sslp}} \mid b \in \mathsf{breaks}^\infty \ \} \\
= & \{\ b + pz - \tfrac{\mathsf{lim}_t[b+pz]}{\mathsf{sslp}} \mid b + p\mathbb{Z} \in \mathsf{breaks}, z \in \mathbb{Z}\ \} \\
= & \{\ b + pz - \tfrac{\mathsf{lim}_t[b] + pz\mathsf{oslp}}{\mathsf{sslp}} \mid b + p\mathbb{Z} \in \mathsf{breaks}, z \in \mathbb{Z}\ \} && \text{by Limit Period (Lem. 13)} \\
& && \text{and Breaks Period (Lem. 14)} \\
= & \{\ b + pz - \tfrac{\mathsf{lim}_t[b]}{\mathsf{sslp}} - \tfrac{pz\mathsf{oslp}}{\mathsf{sslp}} \mid b + p\mathbb{Z} \in \mathsf{breaks}, z \in \mathbb{Z}\ \} \\
= & \{\ b - \tfrac{\mathsf{lim}_t[b]}{\mathsf{sslp}} + pz(1 - \tfrac{\mathsf{oslp}}{\mathsf{sslp}}) \mid b + p\mathbb{Z} \in \mathsf{breaks}, z \in \mathbb{Z}\ \} \\
= & \{\ \mathsf{zero}(b) + pz(1 - \tfrac{\mathsf{oslp}}{\mathsf{sslp}}) \mid b + p\mathbb{Z} \in \mathsf{breaks}, z \in \mathbb{Z}\ \}
\end{aligned}
$$

Thus we know that any $\mathsf{zero}(b)$ for some break b that is within $(\mathsf{distX}^-, \mathsf{distX}^+)$, is in ezero.
Further we notice that we have that if $t \leq \mathsf{distX}^- \leq y$, and $[t, y]$, or $[t + \varepsilon, y]$ is a solution interval, then $[\mathsf{distX}^-, y]$ is a solution interval for any plain term t.
We case split:
case $\mathsf{sslp} = 0$**:**

Then Piecewise Linearity (Lem. 4) and $t[y]\diamond 0$, we know that $t[x']\diamond 0$ for any $x' \in (b^-, b^+)$, hence $[b^- + \varepsilon, y']$ is a solution interval.
If $b^- \leq \mathsf{distX}^-$ then b^- might not be in the elimination set, but in this case we have distX^- in the elimination set with $[\mathsf{distX}^-, y]$ being a solution interval too.

⊣

case $\mathsf{sslp} = 0$.

case $\mathsf{sslp} < 0$ & $\diamond \in \{>, \geq\}$**:**

By Piecewise Linearity (Lem. 4), and Zeros (Lem. 16) we know that $y \lesssim \mathsf{zero}(b^-)$. Let $x' \in (b^-, y]$, then again by Piecewise Linearity (Lem. 4), and Zeros (Lem. 16) we know that $t[x'] \gtrsim 0$. Thus $[b^- + \varepsilon, y]$ is a solution interval.
If $\mathsf{zero}(b^-) \leq \mathsf{distX}^-$ then $\mathsf{zero}(b^-)$ might not be in the elimination set, but in this case we have distX^- in the elimination set with $[\mathsf{distX}^-, y]$ being a solution interval too.

⊣

case $\mathsf{sslp} < 0$ & $\diamond \in \{>, \geq\}$.

case $\mathsf{sslp} > 0$ & $\diamond$ **is** $\gtrsim \in \{>, \geq\}$ & $\mathsf{zero}(b^-) \in (b, b^+)$**:**

Then by Piecewise Linearity (Lem. 4) and Zeros (Lem. 16) we know that $y \gtrsim \mathsf{zero}(b^-)$. Consider any x' such that $\mathsf{zero}(b^-) \lesssim x' \leq y$. By the same lemmas we know that $t[x'] \gtrsim 0$, thus if $\gtrsim$ is $>$ then $[\mathsf{zero}(b^-) + \varepsilon, y]$ is a solution interval, and if $\gtrsim$ is $\leq$ then $[\mathsf{zero}(b^-), y]$ is a solution interval.
If $\mathsf{zero}(b^-) \leq \mathsf{distX}^-$ then $\mathsf{zero}(b^-)$ might not be in the elimination set, but in this case we have distX^- in the elimination set with $[\mathsf{distX}^-, y]$ being a solution interval too.

⊣

case $\mathsf{sslp} > 0$ & $\diamond$ **is** $\gtrsim \in \{>, \geq\}$ & $\mathsf{zero}(b^-) \in (b, b^+)$.

case $\mathsf{sslp} > 0$ & $\diamond$ **is** $\gtrsim \in \{>, \geq\}$ & $\mathsf{zero}(b^-) \notin (b, b^+)$**:**

By Piecewise Linearity (Lem. 4), and Zeros (Lem. 16) we know that $b^+ > y \gtrsim \mathsf{zero}(b^-)$. As $\mathsf{zero}(b^-) \notin (b^-, b^+)$ we know that $b^+ > y > b^- \geq \mathsf{zero}(b^-)$. Consider any $x' \in (b^-, y]$. By Piecewise Linearity (Lem. 4), and Zeros (Lem. 16) we know that $t[x'] > 0$, thus $[b + \varepsilon, y]$ is a solution interval.
If $b^- \leq \mathsf{distX}^-$ then b^- might not be in the elimination set, but in this case we have distX^- in the elimination set with $[\mathsf{distX}^-, y]$ being a solution interval too.

⊣

case $\mathsf{sslp} > 0$ & $\diamond$ **is** $\gtrsim \in \{>, \geq\}$ & $\mathsf{zero}(b^-) \notin (b, b^+)$.

case $\mathsf{sslp} \neq 0$ & $\diamond$ **is** $\not\approx$**:**

If $t[y] \neq 0$ we know that either $t[y] > 0$ or $-t[y] > 0$, which means that we can reason as in the cases before to establich that either $[\mathsf{zero}(b^-) + \varepsilon, y]$, $[b^- + \varepsilon, y]$, or $[\mathsf{distX}^-, y]$ will be a solution interval with lower bound in $\mathsf{elim}(L)$.

⊣

case $\mathsf{sslp} \neq 0$ & $\diamond$ **is** $\not\approx$.

case $\mathsf{sslp} \neq 0$ & $\diamond$ **is** $\approx$**:**

Then by Piecewise Linearity (Lem. 4), and Zeros (Lem. 16), it must be the case that $y = \mathsf{zero}(b)$, thus $[\mathsf{zero}(b), y']$ is a solution interval.

⊣

case $\mathsf{sslp} \neq 0$ & $\diamond$ **is** $\approx$.

case $y \in (\mathsf{distX}^-, \mathsf{distX}^+)$ & $y \notin \mathsf{breaks}^\infty$.

case $\mathsf{oslp} \neq 0$.

case $\mathsf{breaks} \neq \emptyset$.

□

Lemma 18. *Let t be a virtual term and ϕ a conjunction of literals. Then*

$$\mathbb{R} \models \exists x.\mathsf{sInter}(t, x, \phi) \rightarrow \phi[\![t]\!]$$

Proof. Let $\mathcal{I}$ be an arbitrary $\mathbb{R}$-interpreation. We reason in $\mathcal{I}$. We case split on t.

base case $t = t$ is a plain virtual term :

Then $\mathsf{sInter}(t, x, \phi) = t \leq x \land \forall x'(t \leq x' \leq x \to \phi[x'])$. Thus obviously $\phi[\![t]\!]$ holds. ⊣

base case $t = t$ is a plain virtual term .

base case $t = t_0 + \varepsilon$ where t_0 is a plain virtual term:

Then $\mathsf{sInter}(t_0 + \varepsilon, x, \phi) = t_0 < x \land \forall x'.(t_0 < x' \leq x \to \phi[x'])$. Let $L = s \diamond 0$ be any literal of ϕ.

case $\mathsf{breaks} \approx \emptyset$:

By Piecewise Linearity (Lem. 4) we know that there are some b^-, b^+ such that $t_0 \in (b^-, b^+)$, such that s is a linear segment in (b^-, b^+). Further this means there is is some u, with $t_0 < u < min(b^+, x)$.

case $\mathsf{breaks} \approx \emptyset$.

case $\mathsf{breaks} \not\approx \emptyset$:

In this case we choose $u = x$, and we know by Lem. 11 that $s[x'] \approx \mathsf{sslp}_s \cdot x' + \mathsf{dseg}_s(t_0)$.

case $\mathsf{breaks} \not\approx \emptyset$.

Thus in either case we know that for any $x' \in (t_0, u]$ (P_3) $s[x'] \diamond 0$ holds, and that (P_4) $s[x'] \approx \mathsf{sslp}_s \cdot x' + \mathsf{dseg}_s(t_0)$. Further we can see for any such x' that

$$\begin{aligned} s[x'] &\approx \mathsf{sslp}_s \cdot x' + \mathsf{dseg}_s(t_0) && \text{by } P_4 \\ &\approx \mathsf{sslp}_s \cdot x' - \mathsf{sslp}_s \cdot t_0 + \mathsf{lim}_s[t_0] \\ &\approx \mathsf{sslp}_s(x' - t_0) + \mathsf{lim}_s[t_0] \end{aligned}$$

and thus we know that $x' - t_0 > 0$.

We case split on L

case $s \diamond 0$ & $\mathsf{sslp}_s = 0$:

Thus $L[\![t_0]\!] = \mathsf{lim}_s[t_0] \diamond 0$. As $s[x'] \diamond 0$ we know that $\underbrace{\mathsf{sslp}_s}_{\approx 0}(x' - t_0) + \mathsf{lim}_s[t_0] \diamond 0$, which means that $\mathsf{lim}_s[t_0] \diamond 0$. ⊣

case $s \diamond 0$ & $\mathsf{sslp}_s = 0$.

case $s \gtrsim 0$ & $\mathsf{sslp}_s > 0$:

Thus $L[\![t_0]\!] = \mathsf{lim}_s[t_0] \geq 0$.

As $s[x'] \gtrsim 0$ we know that $\underbrace{\mathsf{sslp}_s}_{>0}\underbrace{(x' - t_0)}_{>0} + \mathsf{lim}_s[t_0] \gtrsim 0$, which means that $\mathsf{lim}_s[t_0] \geq 0$. ⊣

case $s \gtrsim 0$ & $\mathsf{sslp}_s > 0$.

case $s \gtrsim 0$ & $\mathsf{sslp}_s < 0$:

Thus $L[\![t_0]\!] = \mathsf{lim}_s[t_0] > 0$. As $s[x'] \gtrsim 0$ we know that $\underbrace{\mathsf{sslp}_s}_{<0}\underbrace{(x' - t_0)}_{>0} + \mathsf{lim}_s[t_0] \gtrsim 0$, which means that $\mathsf{lim}_s[t_0] > 0$.

⊣

case $s \gtrsim 0$ & $\mathsf{sslp}_s < 0$.

case $s \approx 0$ & $\mathsf{sslp}_s \neq 0$:

Thus $L[\![t_0]\!] = \bot$. We will show that this branch leads to a contradiction thus, is unreachable: Consider some $x'' \in (t_0, x')$.

$$\begin{aligned} & s[x'] \approx 0 \approx s[x''] \\ \iff & \mathsf{sslp}_s(x' - t_0) + \mathsf{lim}_s[t_0] \approx \mathsf{sslp}_s(x'' - t_0) + \mathsf{lim}_s[t_0] \\ \iff & x' \approx x'' \end{aligned}$$

which contradicts that $x'' < x'$.

⊣

case $s \approx 0$ & $\mathsf{sslp}_s \neq 0$.

case $s \not\approx 0$ & $\mathsf{sslp} \neq 0$:

Thus $L[\![t_0]\!] = \top$. In this case we need to show that there is always some x such that $[t_0 + \varepsilon, x]$ is a solution interval.
We case spit on $\mathsf{lim}_s[t_0]$:
case $\mathsf{lim}_s[t_0] > 0$ & $\mathsf{sslp}_s > 0$:

Consider any $x' \in (t_0, u]$. We know that $x' - t_0 > 0$ and $\mathsf{sslp}_s > 0$, thus we know that $\mathsf{sslp}_s(x' - t_0) > 0$, thus $s[x'] \approx \mathsf{sslp}_s(x' - t_0) + \mathsf{lim}_s[t_0] > 0$. This means that $[t_0 + \varepsilon, u]$ is a solution interval.

⊣

case $\mathsf{lim}_s[t_0] > 0$ & $\mathsf{sslp}_s > 0$.

case $\mathsf{lim}_s[t_0] > 0$ & $\mathsf{sslp}_s < 0$:

Observe that

$$\mathsf{lim}_s[t_0] > 0 \iff \underbrace{t_0 - \frac{\mathsf{lim}_s[t_0]}{\mathsf{sslp}_s}}_{\mathsf{zero}_s(t_0)} > t_0$$

Let $x \in (t_0, min(u, \mathsf{zero}_s(t_0)))$. Consider any $x' \in (t_0, x]$. We have

$$\begin{aligned} & x' < \mathsf{zero}_s(t_0) \\ \iff & x' < t_0 - \frac{\mathsf{lim}_s[t_0]}{\mathsf{sslp}_s} \\ \iff & \mathsf{sslp}_s(x' - t_0) + \mathsf{lim}_s[t_0] > 0 \\ \iff & s[x'] > 0 && \text{by Lem. 4} \end{aligned}$$

Thus $[t_0 + \varepsilon, x]$ is a solution interval.

⊣

case $\mathsf{lim}_s[t_0] > 0$ & $\mathsf{sslp}_s < 0$.

case $\mathsf{lim}_s[t_0] < 0$ & $\mathsf{sslp}_s < 0$**:**

This can be done analogous to the case where $\mathsf{lim}_s > 0$ & $\mathsf{sslp} > 0$.

⊣

case $\mathsf{lim}_s[t_0] < 0$ & $\mathsf{sslp}_s < 0$.

case $\mathsf{lim}_s[t_0] < 0$ & $\mathsf{sslp}_s > 0$**:**

This can be done analogous to the case where $\mathsf{lim}_s > 0$ & $\mathsf{sslp} < 0$.

⊣

case $\mathsf{lim}_s[t_0] < 0$ & $\mathsf{sslp}_s > 0$.

case $\mathsf{lim}_s[t_0] \approx 0$**:**

Let $x' \in (t_0, u]$. Then we have $\underbrace{\mathsf{sslp}_s(x' - t_0)}_{\not\approx 0} + \underbrace{\mathsf{lim}_s[t_0]}_{\approx 0} \not\approx 0$. This means $[t_0 + \varepsilon, u]$ is a solution interval.

⊣

case $\mathsf{lim}_s[t_0] \approx 0$.

case $s \not\approx 0$ & $\mathsf{sslp} \neq 0$.

base case $t = t_0 + \varepsilon$ where t_0 is a plain virtual term.

base case $t = t_0 - \infty$**:**

Then $\mathsf{slnter}(-\infty, x, \phi) = \forall x'(x' \leq x \to \phi[x'])$, which means by Periodic Literal Induction (Lem. 15) that all periodic liteals of ϕ must be constant true for any value. Further by Limit Value (Lem. 6) all aperiodic literals L must have a truth limit $\mathsf{lim}_L^{-\infty} = \top$. Therefore $\phi[\![t_0 - \infty]\!]$ must hold.

⊣

base case $t = t_0 - \infty$.

base case $t = t_0 + \infty$**:**

Then $\mathsf{slnter}(-\infty, x, \phi) = \bot$, thus this can never be the case.

⊣

base case $t = t_0 + \infty$.

case $t = t_0 + p\mathbb{Z}$**:**

Then $\mathsf{slnter}(t_0 + p\mathbb{Z}, x, \phi) = \exists z \in \mathbb{Z}.\mathsf{slnter}(t_0 + pz, \phi)$ We case split along the definition of virtual substitution Def. 9. Let P and A be the periodic and aperiodic literals of ϕ as in the defintion. Let $t_0 = t' + e\varepsilon$.

case $V1.\forall L \in A.\mathsf{lim}_L^{\pm\infty} = \top$**:**

Suppose $\mathsf{slnter}(t' + e\varepsilon + pz, x, \phi)$ for some $z \in \mathbb{Z}$. Then $\mathsf{slnter}(t' + e\varepsilon + pz, x, \bigwedge P)$ holds as well. Thus there is some $z' \in \mathbb{Z}$ such that $t' + pz + \lambda z' \in [t', t' + \lambda)$. Further as the truth value of all peridoic literals repeats Lem. 5 for λ, it must be the case that $\mathsf{slnter}(t' + e\varepsilon + pz + \lambda z', x, \bigwedge P)$ holds. As $\frac{\lambda}{p} \in \mathbb{Z}$, we know that $pz + \lambda z' \approx pz''$ for some $z \in \mathbb{Z}$. Thus as by Lem. 3 all such values $t' + pz''$ are in $\mathsf{fin}^{\phi}_{t'+p\mathbb{Z}}$, we can reason in the base cases before to get that for some $t'' \in \mathsf{fin}^{\phi}_{t'+p\mathbb{Z}}$ we have $\bigwedge_{L \in P} L[\![t'' + e\varepsilon]\!]$ which implies that $\bigwedge_{L \in P} L[\![t'' + e\varepsilon \pm \infty]\!]$ which in turn means due to the condition V1 that $\bigwedge_{L \in \phi} L[\![t'' + e\varepsilon \pm \infty]\!]$. Thus $\phi[\![t_0 + p\mathbb{Z}]\!]$ holds.

⊣

case $V1.\forall L \in A.\mathsf{lim}_L^{\pm\infty} = \top$.

case $V2.\exists L \in A.L = u \approx 0$:

If for some $z \in \mathbb{Z}$ we have $\mathsf{sInter}(t' + e\varepsilon + pz, x, \phi)$, then as $\mathsf{lim}_{u\approx 0}^{\pm\infty} = \bot$, by Lem. 6 it must be the case that the solution interval $[t' + e\varepsilon + pz, x]$ is within the core interval $I = [\mathsf{distX}_{u\approx 0}^-, \mathsf{distX}_{u\approx 0}^+]$, which means $x \in I$ and $t' + pz \in I$. As by Lem. 3 all such $t' + pz$ are covered by $\mathsf{fin}_{t'+p\mathbb{Z}}^\phi$ we can reason as in the base cases before to get $\bigvee_{t''\in\mathsf{fin}_{t'+p\mathbb{Z}}^\phi} \phi[\![t''+e\varepsilon]\!]$. Thus $\phi[\![t_0 + p\mathbb{Z}]\!]$ holds.

⊣

case $V2.\exists L \in A.L = u \approx 0$.

case $V3$.otherwise:

If for some $z \in \mathbb{Z}$ we have $\mathsf{sInter}(t'+e\varepsilon+pz, x, \phi)$, then there are the following cases:

case $t' + pz \in [\mathsf{distX}_L^-, \mathsf{distX}_L^+]$ for some $L \in A$ with $\mathsf{lim}_L^{-\infty} = \bot$:

Then we can reason as in the case V2.

⊣

case $t' + pz \in [\mathsf{distX}_L^-, \mathsf{distX}_L^+]$ for some $L \in A$ with $\mathsf{lim}_L^{-\infty} = \bot$.

case otherwise:

Then by Lem. 6 $t' + pz$ must be in $(\mathsf{distX}_L^+, \infty)$ for every $L \in A$ with $\mathsf{lim}_L^{-\infty} = \bot$. Note that there must be at least one such L because the opposite was covered by $V1$. Let distX^+ be the maximal of all these distX_L^+. Thus $t' + pz \in (\mathsf{distX}^+, \infty)$. Hence, we have $t'+pz+\lambda z' \in (\mathsf{distX}^+, \mathsf{distX}^+ + \lambda]$ for some $z' \in \mathbb{Z}$. All aperiodic literals are true $t'+pz+\lambda z'$ as it is in $(\mathsf{distX}^+, \infty)$. As all aperiodic literals' truth values repeat Lem. 5 at a period of λ, we also have that they all hold for $t' + pz + \lambda z'$. This means $t' + pz + \lambda z' \approx t' + pz''$ for some $z'' \in \mathbb{Z}$. All such $t' + pz''$ are covered by $\mathsf{fin}_{t'+p\mathbb{Z}}^\phi$ thus we can again reason as in the base cases to get that $\bigvee_{t''\in\mathsf{fin}_{t'+p\mathbb{Z}}^\phi} \phi[\![t' + e\varepsilon + p\mathbb{Z}]\!]$ which means that $\phi[\![t_0 + p\mathbb{Z}]\!]$ holds.

⊣

case otherwise.

case $V3$.otherwise.

case $t = t_0 + p\mathbb{Z}$.

□

Lemma 19. *Let t be some virtual term, and ϕ be a conjunction of* Lira*-literals.*

$$\mathbb{R} \models \phi[\![t]\!] \to \exists x.\phi$$

Proof. We reason in an arbitrary $\mathbb{R}$-interpretation $\mathcal{I}$ and case split on t.

case $t = t_0$ is a plain virtual term :

Then the implication obviously holds.

⊣

case $t = t_0$ is a plain virtual term .

case $t = t_0 + \varepsilon$ where t_0 is a plain virtual term :

Then $\phi[\![t]\!]$ holds for every literal L. Let L be any of these literals. We will show that L is true for all values in some non-empty interval $(t_0, x_L]$. If this is the case then obviously ϕ will be true $(t_0, \min\{x_L \mid L \in \phi\}]$, which measn that the lemma holds.

For this let us first show the properties P_1, and P_2:

lemma P_1:

$$\mathsf{lim}_s[t_0] \geq 0 \ \& \ \mathsf{sslp}_s > 0 \Longrightarrow \exists x_L.\forall x \in (t_0, x_L].s[x] > 0 \qquad (P_1)$$

Consider the interval (t_0, b^+), where $b^+ = \min\{b \in \mathsf{breaks}_t^\infty \mid t_0 < b^+\}$, and any $x \in (t_0, b^+)$.

$$\begin{aligned} & \mathsf{lim}_s[t_0] \geq 0 \\ \Longrightarrow\ & \mathsf{lim}_s[t_0] + \underbrace{\mathsf{sslp}_s(x - t_0)}_{>0} > 0 \\ \Longrightarrow\ & s[x] > 0 && \text{by Piecewise Linearity (Lem. 4)} \end{aligned}$$

Hence we can choose any value in (t_0, b^+) for x_L.

⊣

lemma P_1.

lemma P_2:

$$\mathsf{lim}_s[t_0] > 0 \ \& \ \mathsf{sslp}_s < 0 \Longrightarrow \exists x_L.\forall x \in (t_0, x_L].s[x] > 0 \qquad (P_2)$$

Consider the intervals $(t_0, \mathsf{zero}_s(t_0)]$, and (t_0, b^+), where $b^+ = \min\{b \in \mathsf{breaks}_t^\infty \mid t_0 < b^+\}$, and any $x \in (t_0, b^+)$. The interval $(t_0, \mathsf{zero}_s(t_0)]$ is non-empty:

$$\mathsf{lim}_s[t_0] > 0 \Longleftrightarrow -\frac{\mathsf{lim}_s[t_0]}{\mathsf{sslp}_s} > 0 \Longleftrightarrow \underbrace{t_0 - \frac{\mathsf{lim}_s[t_0]}{\mathsf{sslp}_s}}_{\mathsf{zero}_s(t_0)} > t_0$$

Thus let x_L be any value in the intersection of the two intervals. For any value $x \in (t_0, x_L)$ we get

$$\begin{aligned} & x < \overbrace{t_0 - \frac{\mathsf{lim}_s[t_0]}{\mathsf{sslp}_s}}^{\mathsf{zero}_s(t_0)} \\ \Longleftrightarrow\ & \mathsf{sslp}_s \cdot (x - t_0) + \mathsf{lim}_s[t_0] > 0 \\ \Longleftrightarrow\ & s[x] > 0 && \text{by Piecewise Linearity (Lem. 4)} \end{aligned}$$

⊣

lemma P_2.

Thus let us now case split on L:

case $s \diamond 0 \ \& \ \mathsf{sslp}_s = 0$:

Consider the interval (t_0, b^+), where $b^+ = \min\{b \in \mathsf{breaks}_t^\infty \mid t_0 < b^+\}$, and any $x \in (t_0, b^+)$.

$$\begin{aligned} &\overbrace{\mathsf{lim}_s[t_0] \diamond 0}^{L[\![t_0+\varepsilon]\!]} \\ \Longrightarrow\ &\mathsf{lim}_s[t_0] + \underbrace{\mathsf{sslp}_s(x - t_0)}_{\approx 0} \diamond 0 \\ \Longrightarrow\ &s[x] \diamond 0 \qquad \text{by Piecewise Linearity (Lem. 4)} \end{aligned}$$

Hence we can choose any value in (t_0, b^+) for x_L.

⊣

case $s \diamond 0$ & $\mathsf{sslp}_s = 0$.

case $s \gtrsim 0$**:**

case $\mathsf{sslp}_s > 0$**:**

Then $L[\![t_0 + \varepsilon]\!] = \mathsf{lim}_s[t_0] \geq 0$, thus we can use P_1, to get (t_0, x_L).

⊣

case $\mathsf{sslp}_s > 0$.

case $\mathsf{sslp}_s < 0$**:**

Then $L[\![t_0 + \varepsilon]\!] = \mathsf{lim}_s[t_0] > 0$, thus we can use P_2, to get (t_0, x_L).

⊣

case $\mathsf{sslp}_s < 0$.

case $s \gtrsim 0$.

case $s \approx 0$**:**

Then it must be the case that $\mathsf{sslp}_s = 0$, thus we already handled this case.

⊣

case $s \approx 0$.

case $s \not\approx 0$**:**

We know that $\mathsf{sslp}_s \neq 0$. Thus there are 4 cases in which we can use P_1 and P_2 to find the right interval $(t_0, x_L]$:

- if $\mathsf{sslp}_s < 0$ & $\mathsf{lim}_s[t_0] > 0$ we use P_2
- if $\mathsf{sslp}_s < 0$ & $-\mathsf{lim}_s[t_0] \geq 0$ we use P_1
- if $\mathsf{sslp}_s > 0$ & $\mathsf{lim}_s[t_0] \geq 0$ we use P_1
- if $\mathsf{sslp}_s > 0$ & $-\mathsf{lim}_s[t_0] > 0$ we use P_2

⊣

case $s \not\approx 0$.

case $t = t_0 + \varepsilon$ **where** t_0 **is a plain virtual term** .

case $t = t_0 - \infty$**:**

If $\phi[\![t_0 - \infty]\!]$ holds then by Virtual Substitution (Def. 9) for all aperiodic literals A of ϕ it must be the case that $\mathsf{lim}_L^{-\infty} = \top$. Further for every periodic literals P we know that $P[\![t_0]\!]$ holds. Thus by the previous cases of this lemma we know that there is some value x_0 such that all periodic literals hold for x_0. Let distX^- be the minimum of all distX_A^- of all aperiodic literals A. Choose some z such that $x_0 + \lambda z < \mathsf{distX}^-$. All aperiodic literals are true $x_0 + \lambda$ due to Limit Value (Lem. 6), and all peridic literals will be true due to Periodic Literals (Lem. 5).

⊣

case $t = t_0 - \infty$.

case $t = t_0 + \infty$**:**

Can be proven in the same way as the case where $t = t_0 - \infty$.

⊣

case $t = t_0 + \infty$.

case $t = t_0 + p\mathbb{Z}$**:**

If $\phi[\![t_0 + p\mathbb{Z}]\!]$ holds, then $\phi[\![s + e\varepsilon]\!]$ holds for some $s \in \mathsf{fin}_\phi^{t_0+p\mathbb{Z}}$. Thus we can reason as in the base cases of this lemma we get that ϕ holds for some x.

⊣

case $t = t_0 + p\mathbb{Z}$.

□

Lemma 20 (Solution Interval Intersection)**.** *Let x be a variable, s, t be virtual terms, and ϕ, ψ be formulas, and $\mathcal{I}$ be an $\mathbb{R}$-interpretation. If $[s, x]$ is a solution interval of ϕ, and $[t, x]$ is a solution interval of ψ, with respect to $\mathcal{I}$ then either of the two is a solution interval of $\phi \land \psi$. More formally:*

$$\mathbb{R} \models \mathsf{sInter}(s, x, \phi) \land \mathsf{sInter}(t, x, \phi) \to \mathsf{sInter}(s, x, \phi \land \psi) \lor \mathsf{sInter}(t, x, \phi \land \psi)$$

Proof. This can be easily seen by just unfolding definitions. □

Theorem 1 (Quantifier Elimination; repeated)**.** *Let ϕ be a non-empty conjunction of* Lira*-literals.*

$$\mathbb{R} \models \exists x.\phi \leftrightarrow \bigvee_{t \in \mathsf{elim}(\phi)} \phi[\![t]\!]$$

Proof. We reason in an arbitrary $\mathbb{R}$-interpretation $\mathcal{I}$. Suppose $\mathcal{I} \models \phi$. Then $\mathcal{I} \models L$ every $L \in \phi$. From Lem. 17 we get terms v_L such that $\mathsf{sInter}(v_L, x, L)$. By lemma Lem. 20 we get some v sch that $\mathsf{sInter}(v, x, \phi)$. Thus by Lem. 18 we know that $\phi[\![x /\!/ v]\!]$ holds.

Now suppose that $\phi[\![t]\!]$ holds for some $t \in \mathsf{elim}(\phi)$. Then by Lem. 19 we know that $\exists x.\phi$ holds. □

Lemma 7 (repeated)**.** *Let ϕ be a conjunction of literals and v be a virtual term with $\mathbb{Z}(v) = 0$. Our function lemma_ϕ satisfies the following properties:*

1. $\neg\phi[\![x /\!/ v]\!] \to \forall x(\phi \to \mathsf{lemma}_\phi(x \not\approx v))$. *(soundness)*
2. $\neg\mathsf{lemma}_\phi[\![x /\!/ v]\!]$. *(completeness)*

Proof. We first show soundness: In the following let s, t be terms. We case split on $v \not\approx 0$, along the defintion of lemma_ϕ Def. 12.

case $t \not\approx 0$**:**

Then $\mathsf{lemma}_\phi(v \not\approx 0) = x \not\approx t$. In this case the lemma obviously follows, as $\phi[x/v] \leftrightarrow \exists x(\phi[x] \land x \approx v)$.

⊣

case $t \not\approx 0$.

case $s + \varepsilon \not\approx 0$**:**

Then $\mathsf{lemma}_\phi(v \not\approx 0) = \neg\mathsf{inFalseInterval}^F_{s+\varepsilon}(x)$. Assume $\neg\phi[\![s+\varepsilon]\!]$. To proof the lemma need now show that for any x if $\mathsf{inFalseInterval}^F_{s+\varepsilon}(x)$ holds, then $\phi[x]$ is false.
Let x be arbitrary, such that $\mathsf{inFalseInterval}^F_{s+\varepsilon}(x)$ holds. As $\phi[\![s+\varepsilon]\!]$ is false there must be some literal $L = t \diamond 0$ of ϕ such that $L[\![s+\varepsilon]\!]$ is false.

case $\mathsf{breaks} = \emptyset \wedge \mathsf{sslp} = 0$**:**

Then the value of t is constant wrt. x, hence L is false for any x.

⊣

case $\mathsf{breaks} = \emptyset \wedge \mathsf{sslp} = 0$.

case $\mathsf{breaks} = \emptyset$**:**

As $\mathsf{breaks} = \emptyset$ by Lem. 11 we know that t is a linear function with zero $\mathsf{zero}(()0)$.
We case split on $t \diamond 0$.

case $t \approx 0$**:**

The only value where $t \approx 0$ can be true is $\mathsf{zero}(0)$, and $\mathsf{zero}(0)$ is in $\mathsf{nxt}^\top_{t\approx 0} s + \varepsilon$. Thus we know by the defintion of $\mathsf{inFalseInterval}_{s+\varepsilon}(x)$ that either $x < \mathsf{zero}(0)$ or $\mathsf{zero}(0) \leq t < x$, thus $t \approx 0$ is false for any such x.

case $t \approx 0$.

case $t \not\approx 0$**:**

This cannot be the case as then $L[\![s+\varepsilon]\!] = \top$, thus it cannot be false as assumed before.

⊣

case $t \not\approx 0$.

case $t > 0$ & $\mathsf{sslp} > 0$**:**

Then we got $\neg L[\![s+\varepsilon]\!] = \mathsf{lim}_s[t] < 0 = s[t] < 0$.
We have $\mathsf{zero}(0) + \varepsilon$ in $\mathsf{nxt}^\top_{t>0}(s+\varepsilon)$, and therefore by the defintion of the false interval we know that there are two cases:

case $\neg(t + \varepsilon \lessdot \mathsf{zero}(0) + \varepsilon)$**:**

If we expand the defintions this means that $t \geq \mathsf{zero}(0)$, Which means $t[s] \geq 0$, which contradicts the assumption that $\neg L[\![s+\varepsilon]\!]$.

⊣

case $\neg(t + \varepsilon \lessdot \mathsf{zero}(0) + \varepsilon)$.

case $x \lessdot \mathsf{zero}(0) + \varepsilon$**:**

If we expand the defintions this means that $x \leq \mathsf{zero}(0)$. As t is a linear term this obviously means that $t > 0$ will be false for t.

⊣

case $x \lessdot \mathsf{zero}(0) + \varepsilon$.

case $t > 0$ & $\mathsf{sslp} > 0$.

case $t \geq 0$ & $\mathsf{sslp} > 0$**:**

Then we got $\neg L[\![s+\varepsilon]\!] = \mathsf{lim}_s[t] < 0 = s[t] < 0$.

We have $\mathsf{zero}(0)$ in $\mathsf{nxt}^{\top}_{t\geq 0}(s+\varepsilon)$, and therefore by the defintion of the false interval we know that there are two cases:

case $\neg(t+\varepsilon \lessdot \mathsf{zero}(0))$**:**

If we expand the defintions this means that $t > \mathsf{zero}(0)$, Which means $t[s] > 0$, which contradicts the assumption that $\neg L[\![s+\varepsilon]\!]$.

⊣

case $\neg(t+\varepsilon \lessdot \mathsf{zero}(0))$.

case $x \lessdot \mathsf{zero}(0)$**:**

If we expand the defintions this means that $x < \mathsf{zero}(0)$. As t is a linear term this obviously means that $t \geq 0$ will be false for t.

⊣

case $x \lessdot \mathsf{zero}(0)$.

case $t \geq 0$ & $\mathsf{sslp} > 0$.

case $t \gtrsim 0$ & $\mathsf{sslp} < 0$**:**

Then we got $\neg L[\![s+\varepsilon]\!] = \mathsf{lim}_s[t] \leq 0 = s[t] \leq 0$.
In order for $s[t] \leq 0$ to be true it must be the case that $\mathsf{zero}(0) \leq t$, thus as $t < x$ we have $s[x] \leq 0$ as well.

case $t \gtrsim 0$ & $\mathsf{sslp} < 0$.

case $\mathsf{breaks} = \emptyset$.

case $\mathsf{breaks} \neq \emptyset$**:**

Let $b \in \mathsf{breaks}^{\infty}$ be the least discontinuity greater than s. It is easy to see that it must be the case that $b \in \mathsf{nextBreak}(s)$. By Piecewise Linearity (Lem. 4) we know that within the interval (s, b), is equal to a linear function $t[x] \approx \mathsf{sslp}(x-s) + \mathsf{lim}_t[s]$.

case $\mathsf{sslp} = 0$**:**

We have $b \in \mathsf{nxt}^{\top}_{t\diamond 0}(s+\varepsilon)$, and $s < b$, thus by the defintion of the false interval we know that $s < x < b$, hence as $\neg L[\![s+\varepsilon /\!/ =]\!]\neg(\mathsf{lim}_t[s] \diamond 0)$, and $t[x] \approx \underbrace{\mathsf{sslp}(x-s)}_{0} + \mathsf{lim}_t[s]$ we know that L does not hold for x.

⊣

case $\mathsf{sslp} = 0$.

case $\mathsf{sslp} \neq 0$**:**

We case split on $t \diamond 0$.

case $t \approx 0$**:**

Then we have $\neg L[\![s+\varepsilon]\!] = \top$, and $\mathsf{zero}(s), b \in \mathsf{nxt}^{\top}_{t\diamond 0}(s+\varepsilon)$.

case $s < \mathsf{zero}(s)$**:**

Then by the definition of the false interval we have that $x < \mathsf{zero}(s)$, thus as t is a line segment in (s, b) and $x \in (s, b)$, L does not hold for x.

⊣

case $s < \mathsf{zero}(s)$.

case $\mathsf{zero}(s) \leq s$**:**

As $s < x$ this means that as t is a line segment in (s, b) that L cannot hold for x.

⊣

case $\mathsf{zero}(s) \leq s$.

case $t \approx 0$.

case $t \not\approx 0$**:**

Then we have $\neg L[\![s + \varepsilon]\!] = \bot$. This means this case is not reachable.

⊣

case $t \not\approx 0$.

case $t \geq 0$ & $\mathsf{sslp} > 0$**:**

Then we have $\neg L[\![s + \varepsilon]\!] = \mathsf{lim}_t[s] < 0$, and $\mathsf{zero}(s), b \in \mathsf{nxt}^{\top}_{t \diamond 0}(s + \varepsilon)$.

case $s < \mathsf{zero}(s)$**:**

Then by the definition of the false interval we have that $x < \mathsf{zero}(s)$, thus as t is a line segment with positive slope in (s, b) and $x \in (s, b)$, L does not hold for x.

⊣

case $s < \mathsf{zero}(s)$.

case $\mathsf{zero}(s) \leq s$**:**

This case is unreachable as we established $\mathsf{lim}_t[s] < 0$.

⊣

case $\mathsf{zero}(s) \leq s$.

case $t \geq 0$ & $\mathsf{sslp} > 0$.

case $t > 0$ & $\mathsf{sslp} > 0$**:**

Then we have $\neg L[\![s + \varepsilon]\!] = \mathsf{lim}_t[s] < 0$, and $\mathsf{zero}(s) + \varepsilon, b \in \mathsf{nxt}^{\top}_{t \diamond 0}(s + \varepsilon)$.

case $s < \mathsf{zero}(s)$**:**

Then by the definition of the false interval we have that $x \leq \mathsf{zero}(s)$, thus as t is a line segment with positive slope in (s, b) and $x \in (s, b)$, L does not hold for x.

⊣

case $s < \mathsf{zero}(s)$.

case $\mathsf{zero}(s) \leq s$**:**

This case is unreachable as we established $\mathsf{lim}_t[s] < 0$.

⊣

case $\mathsf{zero}(s) \leq s$.

case $t > 0$ & $\mathsf{sslp} > 0$.

case $t \gtrsim 0$ & $\mathsf{sslp} < 0$**:**

Then we have $\neg L[\![s + \varepsilon]\!] = \mathsf{lim}_t[s] \leq 0$ and $b \in \mathsf{nxt}^{\top}_{t \diamond 0}(s + \varepsilon)$.
As $\mathsf{lim}_t[s] \leq 0$, and $\mathsf{sslp} < 0$, L must be false for all $x \in (s, b)$.

⊣

case $t \gtrsim 0$ & $\mathsf{sslp} < 0$.

case $\mathsf{sslp} \neq 0$.

case $\mathsf{breaks} \neq \emptyset$.

As L is false for any x such that $\mathsf{inFalseInterval}^{\phi}_{s+\varepsilon}(x)$ so will be ϕ. Thus the lemma holds.

⊣

case $s + \varepsilon \not\approx 0$.

case $t + e\varepsilon + \infty \not\approx 0$ & $\mathsf{lim}^{+\infty}_{L} = \bot$ **for some** $L \in A$ **:**

Then $\mathsf{lemma}_{\phi}(v \not\approx 0) = x \leq \mathsf{distX}^{+}_{L}$ Let x be arbitrary such that $\neg\mathsf{lemma}_{\phi}(v \not\approx 0) \iff \mathsf{distX}^{+}_{L} < x$ holds for x. By Lem. 6 we know that $L[x]$ does not hold, thus $\phi[x]$ does not hold either.

⊣

case $t + e\varepsilon + \infty \not\approx 0$ & $\mathsf{lim}^{+\infty}_{L} = \bot$ for some $L \in A$.

case $t + e\varepsilon - \infty \not\approx 0$ & $\mathsf{lim}^{-\infty}_{L} = \bot$ **for some** $L \in A$ **:**

Can be shown analgous to the case before.

⊣

case $t + e\varepsilon - \infty \not\approx 0$ & $\mathsf{lim}^{-\infty}_{L} = \bot$ for some $L \in A$.

case $t \pm \infty \not\approx 0$ & $\forall i \in A.\mathsf{lim}^{+\infty}_{L} = \mathsf{lim}^{-\infty}_{L} = \top$**:**

Then $\mathsf{lemma}_{\phi}(v \not\approx 0) = \mathsf{rem}_{\lambda}(x) \not\approx \mathsf{rem}_{\lambda}(t)$ Assume $\neg\phi[\![t \pm \infty]\!]$. As all aperiodic literals L_a have $\mathsf{lim}^{\pm\infty}_{L_a} = \top$, there must be some periodic literal L that is false. Thus by the defintion the virtual substitution we must have that $\neg L[\![t]\!]$. Assume $\mathsf{rem}_{\lambda}(x) \not\approx \mathsf{rem}_{\lambda}(t) = x - \lambda\mathsf{quot}_{\lambda}(x) \approx t - \lambda\mathsf{quot}_{\lambda}(t)$ which means $x \approx t + z\lambda$. As the truth value of λ repeats periodically Lem. 5, this means that L does not hold at x, thus ϕ does not do so either.

⊣

case $t \pm \infty \not\approx 0$ & $\forall i \in A.\mathsf{lim}^{+\infty}_{L} = \mathsf{lim}^{-\infty}_{L} = \top$.

case $t + \varepsilon \pm \infty \not\approx 0$ & $\forall i \in A.\mathsf{lim}^{+\infty}_{L} = \mathsf{lim}^{-\infty}_{L} = \top$**:**

Then $\mathsf{lemma}_{\phi}(v \not\approx 0) = \neg\mathsf{inFalseInterval}^{F}_{t+\lambda(\mathsf{quot}_{\lambda}(x)-\mathsf{quot}_{\lambda}(t))+\varepsilon}(x)$. Assume $\neg\phi[\![t + \varepsilon \pm \infty]\!]$. As all aperiodic literals L_a have $\mathsf{lim}^{\pm\infty}_{L_a} = \top$, there must be some periodic literal L that is false. Thus we must have that $\neg L[\![t + \varepsilon]\!]$, and as L is periodic by Lem. 5 we know that $\neg L[\![t + \lambda(\mathsf{quot}_{\lambda}(x) - \mathsf{quot}_{\lambda}(t)) + \varepsilon]\!]$ also holds.
Assume $\mathsf{inFalseInterval}^{\phi}_{t+\lambda(\mathsf{quot}_{\lambda}(x)-\mathsf{quot}_{\lambda}(t))+\varepsilon}(x)$. We can reason as in the case before $v \not\approx 0 = t + \varepsilon \not\approx 0$, to obtain that L does not hold for x.

⊣

case $t + \varepsilon \pm \infty \not\approx 0$ & $\forall i \in A.\mathsf{lim}^{+\infty}_{L} = \mathsf{lim}^{-\infty}_{L} = \top$.

Now that we have shown soundness let us show completeness:

case $t \not\approx 0$**:**

Then $\mathsf{lemma}_{\phi}(v \not\approx 0) = x \not\approx t$. It is obviously the case that $\neg(x \not\approx t)[\![t]\!]$.

⊣

case $t \not\approx 0$.

case $s + \varepsilon \not\approx 0$**:**

Then $\mathsf{lemma}_{\phi}(v \not\approx 0) = \neg\mathsf{inFalseInterval}^{F}_{s+\varepsilon}(x)$. This holds as the first literal of $\mathsf{inFalseInterval}^{F}_{s+\varepsilon}(x) = s < x$, which obviously holds for $s+\varepsilon$. All other conjuncts are of the shape $s + \varepsilon \lessdot e \to x \lessdot e$. Thus the gaurd makes sure that these conjuncts will hold for $s + \varepsilon$.

⊣

case $s + \varepsilon \not\approx 0$.

case $t + e\varepsilon + \infty \not\approx 0$ & $\mathsf{lim}^{+\infty}_{L} = \bot$ **for some** $L \in A$ **:**

This case obviously holds by straight unfolding the defintions.

⊣

case $t + e\varepsilon + \infty \not\approx 0$ & $\mathsf{lim}_L^{+\infty} = \bot$ **for some** $L \in A$ **.**

case $t + e\varepsilon - \infty \not\approx 0$ & $\mathsf{lim}_L^{-\infty} = \bot$ **for some** $L \in A$ **:**

This case obviously holds by straight unfolding the defintions.

⊣

case $t + e\varepsilon - \infty \not\approx 0$ & $\mathsf{lim}_L^{-\infty} = \bot$ **for some** $L \in A$ **.**

case $t \pm \infty \not\approx 0$ & $\forall i \in A.\mathsf{lim}_L^{+\infty} = \mathsf{lim}_L^{-\infty} = \top$**:**

Then $\mathsf{lemma}_\phi(v \not\approx 0) = \mathsf{rem}_\lambda(x) \not\approx \mathsf{rem}_\lambda(t) =: lem$. As lem is periodic, by defintion $lem[\![t \pm \infty]\!] = lem[\![t]\!]$. Therefore obviously $\neg lem[\![t]\!]$ holds.

⊣

case $t \pm \infty \not\approx 0$ & $\forall i \in A.\mathsf{lim}_L^{+\infty} = \mathsf{lim}_L^{-\infty} = \top$.

case $t + \varepsilon \pm \infty \not\approx 0$ & $\forall i \in A.\mathsf{lim}_L^{+\infty} = \mathsf{lim}_L^{-\infty} = \top$**:**

Then $\mathsf{lemma}_\phi(v \not\approx 0) = \neg\mathsf{inFalseInterval}^F_{t+\lambda(\mathsf{quot}_\lambda(x)-\mathsf{quot}_\lambda(t))+\varepsilon}(x) = lem$.
This holds as the first literal of $\mathsf{inFalseInterval}^F_{t+\lambda(\mathsf{quot}_\lambda(x)-\mathsf{quot}_\lambda(t))+\varepsilon}(x) = t + \lambda(\mathsf{quot}_\lambda(x) - \mathsf{quot}_\lambda(t)) < x$, which obviously holds for $t + \varepsilon$. All other conjuncts are of the shape $t + \lambda(\mathsf{quot}_\lambda(x) - \mathsf{quot}_\lambda(t)) + \varepsilon \lessdot e \to x \lessdot e$. Thus the gaurd makes sure that these conjuncts will hold for $t + \varepsilon$. Thus we know that $lem[\![t + \varepsilon]\!]$ is false. Further notice that all the literals in the lem are periodic, hence we get $lem[\![t + \varepsilon]\!] = lem[\![t + \varepsilon \pm \infty]\!]$ is false.

⊣

case $t + \varepsilon \pm \infty \not\approx 0$ & $\forall i \in A.\mathsf{lim}_L^{+\infty} = \mathsf{lim}_L^{-\infty} = \top$.

□

Lemma 8 (repeated)**.** *For any state* (F, S, L)*,* $(F, S \mid x_{k+1} \leftarrow \bot, L)$*, or* $(F, S \mid x_{k+1} \leftarrow ?, L)$*, where* $S = \langle x_1 \leftarrow \langle t_1, J_1\rangle \mid \ldots \mid x_k \leftarrow \langle t_k, J_k\rangle\rangle$

$$\mathbb{R} \models \forall\neg(F/S) \to \forall(F \to \bigvee_{i=1}^{k} \mathsf{lemma}_F(x_i \not\approx t_i))$$

Proof. Apply induction on k, and use Lem. 7.1. □