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Stress field of a single screw dislocation in a plate

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ABSTRACT

Study of dislocations is very important in material science because it helps us to predict the mechanical behavior of metals in the plastic regime. In past studies, scientists and researchers have shown the analytic solutions of stress field in the case of infinite and semi-infinite medium. In this article, we study the stress field of a single screw dislocation in a finite thickness plate considering the presence of image or virtual dislocations. The solution is verified against known or expected results.

Keywords:
Dislocations, Screw Dislocations, Image Dislocations, Stress field

Introduction

Numerous research has been conducted based on dislocation theory[1] to characterize material properties. Using the concept of dislocation in materials, Koehler[2] explained the plastic deformation of materials. Hull and Bacon[3] as well as Weertman and Weertman[1] described the stress field of a positive screw dislocation in an infinite material. This stress field solution is not applicable to a finite plate which has traction-free surfaces.

In this article, we ensure the surfaces in a dislocated plate are traction-free by adding fictitious or virtual image dislocations on both sides of the plate. We formulate the coordinates of the image dislocations and use their stress field as a correcting term to be added to the infinite-material solution. Such superposition ensures zero traction on the free surfaces and therefore represent the correct stress solution at any material point in the plate. The image dislocations are added incrementally till an infinite number of them. Results are presented showing how the correct superposed solution differ in behavior and quantitatively from the infinite-material solution. Results are also shown for solution verification.

Theory/Solution Development

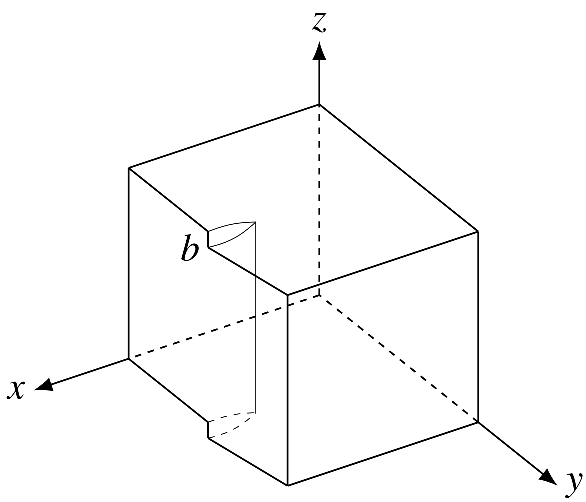
We know in linear elasticity[4], the ij-th component of the small strain tensor is given by,

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial j} + \frac{\partial u_j}{\partial i}\right) \tag{1}$$

and Hooke's law for isotropic material,

$$\sigma_{ij} = \lambda\left(e_{xx} + e_{yy} + e_{zz}\right)\delta_{ij} + 2Ge_{ij} \tag{2}$$

where $i = x, y, z$, $j = x, y, z$ and λ, G are Lamé constant.



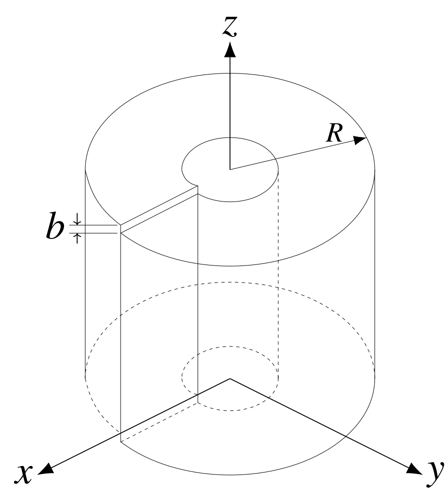


Figure 1: Screw dislocation (schematic) shown in Cartesian and Cylindrical coordinate system.

From fig. 1 we see there is no deformation along x, y direction i.e. $u_x = 0$, $u_y = 0$ and $u_z = \frac{b}{2\pi}\tan^{-1}(y/x)$. Now from eq. (1) we write,

$$e_{xx} = e_{yy} = e_{zz} = e_{xy} = e_{yx} = 0 \tag{3a}$$

$$e_{xz} = e_{zx} = -\frac{b}{4\pi}\frac{y}{x^2+y^2} \tag{3b}$$

$$e_{yz} = e_{zy} = \frac{b}{4\pi}\frac{x}{x^2+y^2} \tag{3c}$$

(3d)

No from eqs. (2) and (3) we can write,

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = \sigma_{yx} = 0 \tag{4a}$$

$$\sigma_{xz} = \sigma_{zx} = 2Ge_{xz} = 2Ge_{zx} = -\frac{Gb}{2\pi}\frac{y}{x^2+y^2} \tag{4b}$$

$$\sigma_{yz} = \sigma_{zy} = 2Ge_{yz} = 2Ge_{zy} = \frac{Gb}{2\pi}\frac{x}{x^2+y^2} \tag{4c}$$

Equation (4) expresses the stress field in the presence of a positive screw dislocation at the origin. We rewrite eq. (4b) as

$$\sigma_{xz} = \sigma_{zx} = -\frac{G(b)}{2\pi}\frac{y}{((x-D_x)^2 + (y-D_y)^2)} \tag{5}$$

where, (D_x, D_y) are the coordinates of the dislocation. We set our origin at the positive dislocation $\S_0$ meaning the coordinates $(D_x, D_y) \equiv (0,0)$ for $\S_0$ and (x, y) are the coordinates of any point P in the thin plate. See fig. 2.

Now the stress on surface 1 ($x \equiv -a$) due to $\S_0$ is,

$$\sigma_{xz} = \sigma_{zx} = -\frac{G(b)}{2\pi}\frac{y}{(a^2 + y^2)} \tag{6}$$

and on surface 2 ($x \equiv d - a$) is

$$\sigma_{xz} = \sigma_{zx} = -\frac{G(b)}{2\pi}\frac{y}{((d-a)^2 + y^2)} \tag{7}$$

Surfaces 1 and 2 should be traction free surfaces, i.e. σ_{xx}, σ_{xy} and σ_{xz} should be zero at these surfaces.

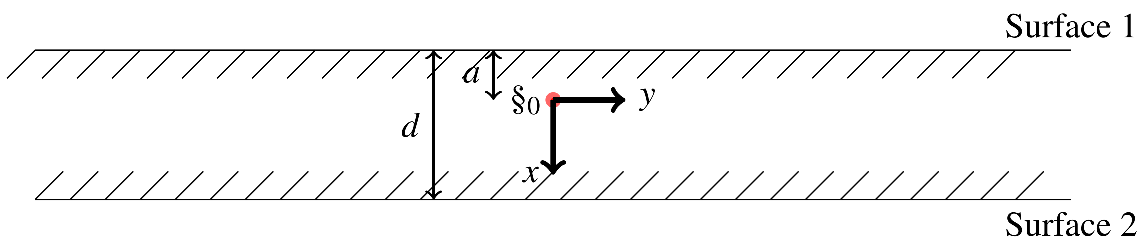


Figure 2: Positive screw dislocation on a thin plate

From eq. (4a) we find $\sigma_{xx} = \sigma_{xy} = 0$ and in eq. (4b) $\sigma_{xz} = \sigma_{zx} \neq 0$ on the free surfaces where $y \neq 0$. But we see σ_{xz} and σ_{zx} should be zero to satisfy the condition of traction free surface. To ensure surfaces 1 and 2 to be traction free (i.e. $\sigma_{xz} = \sigma_{zx} = 0$) we add two fictitious negative dislocations $\reflectbox{\S}_1$ at a distance a from surface 1 (outward) and $\reflectbox{\S}_2$ at a distance $d - a$ from surface 2 (outward) so the net stress would be zero on the surfaces. Note that the symbol $\reflectbox{\S}$ is used to represent negative screw dislocations and the symbol $\S$ is used to represent positive screw dislocations. Now the total stress on surface 1 due to dislocations $\S_0$ and $\reflectbox{\S}_1$ is

$$\begin{aligned}\sigma_{xz} = \sigma_{zx} &= -\frac{G(b)}{2\pi}\frac{y}{((-a)^2 + y^2)} - \frac{G(-b)}{2\pi}\frac{y}{((-a-(-2a))^2 + y^2)} \\ &= -\frac{Gb}{2\pi}\frac{y}{(a^2 + y^2)} + \frac{Gb}{2\pi}\frac{y}{(a^2 + y^2)} \\ &= 0\end{aligned} \tag{8}$$

Similarly the total stress on surface 2 due to dislocations $\S_0$ and $\reflectbox{$\S$}_2$ is

$$\begin{aligned}\sigma_{xz} = \sigma_{zx} &= -\frac{G(b)}{2\pi}\frac{y}{((d-a)^2+y^2)} - \frac{G(-b)}{2\pi}\frac{y}{((d-a-2(d-a))^2+y^2)} \\ &= -\frac{Gb}{2\pi}\frac{y}{((d-a)^2+y^2)} + \frac{Gb}{2\pi}\frac{y}{((d-a)^2+y^2)} \\ &= 0\end{aligned} \tag{9}$$

But dislocation $\reflectbox{$\S$}_1$ causes stress on surface 2 and $\reflectbox{$\S$}_2$ on surface 1. Now the total stress σ_{zx} on surface 1 due to $\S_0$, $\reflectbox{$\S$}_1$ and $\reflectbox{$\S$}_2$

$$\begin{aligned}\sigma_{xz} = \sigma_{zx} &= -\frac{G(b)}{2\pi}\frac{y}{((-a)^2+y^2)} - \frac{G(-b)}{2\pi}\frac{y}{((-a-(-2a))^2+y^2)} - \frac{G(-b)}{2\pi}\frac{y}{((-a-2(d-a))^2+y^2)} \\ &= -\frac{Gb}{2\pi}\frac{y}{(a^2+y^2)} + \frac{Gb}{2\pi}\frac{y}{(a^2+y^2)} + \frac{Gb}{2\pi}\frac{y}{((-2d+a)^2+y^2)} \\ &= \frac{Gb}{2\pi}\frac{y}{((-2d+a)^2+y^2)}\end{aligned} \tag{10}$$

Similarly the total stress on surface 2 due to dislocations $\S_0$, $\reflectbox{$\S$}_1$ and $\reflectbox{$\S$}_2$ is

$$\begin{aligned}\sigma_{xz} = \sigma_{zx} &= -\frac{G(b)}{2\pi}\frac{y}{((d-a)^2+y^2)} - \frac{G(-b)}{2\pi}\frac{y}{((d-a-2(-a))^2+y^2)} - \frac{G(-b)}{2\pi}\frac{y}{((d-a-2(d-a))^2+y^2)} \\ &= -\frac{Gb}{2\pi}\frac{y}{((d-a)^2+y^2)} + \frac{Gb}{2\pi}\frac{y}{((d+a)^2+y^2)} + \frac{Gb}{2\pi}\frac{y}{((d-a)^2+y^2)} \\ &= \frac{Gb}{2\pi}\frac{y}{((d+a)^2+y^2)}\end{aligned} \tag{11}$$

Again we can see the stress $\sigma_{zx} \neq 0$ on the surfaces. So we again add two positive dislocations $\S_3$ at a distance $2d-a$ from surface 1 (outward) and $\S_4$ at a distance $d+a$ from surface 2 (outward) so the net stress would be zero on the surfaces. Now the total stress on surface 1 due to dislocations $\S_0$, $\reflectbox{$\S$}_1$, $\reflectbox{$\S$}_2$ and $\S_3$ is

$$\begin{aligned}\sigma_{xz} = \sigma_{zx} &= \frac{Gb}{2\pi}\frac{y}{((2d-a)^2+y^2)} - \frac{Gb}{2\pi}\frac{y}{((-a-(-2d))^2+y^2)} \\ &= \frac{Gb}{2\pi}\frac{y}{((2d-a)^2+y^2)} - \frac{Gb}{2\pi}\frac{y}{((2d-a)^2+y^2)} \\ &= 0\end{aligned} \tag{12}$$

Similarly the total stress on surface 2 due to dislocations $\S_0$, $\reflectbox{$\S$}_1$, $\reflectbox{$\S$}_2$ and $\S_4$ is

$$\begin{aligned}\sigma_{xz} = \sigma_{zx} &= \frac{Gb}{2\pi}\frac{y}{((d+a)^2+y^2)} - \frac{Gb}{2\pi}\frac{y}{((d-a)-2d)^2+y^2)} \\ &= \frac{Gb}{2\pi}\frac{y}{((d+a)^2+y^2)} - \frac{Gb}{2\pi}\frac{y}{((d+a)^2+y^2)} \\ &= 0\end{aligned} \tag{13}$$

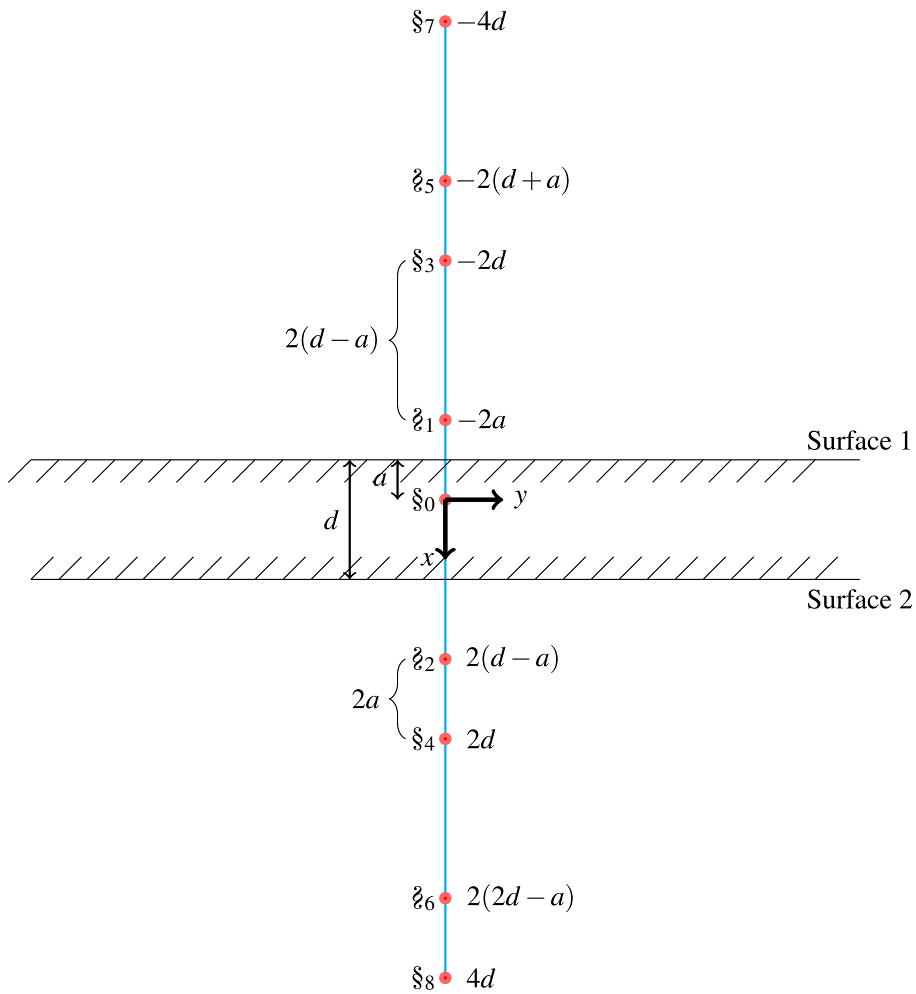


Figure 3: Locations of image dislocations

Again dislocation $\S_3$ causes stress on surface 2 and $\S_4$ on surface 1. Now the total stress σ_{zx} on surface 1 due to $\S_0$, $\S_1$, $\S_2$, $\S_3$ and $\S_4$ is

$$\begin{aligned} \sigma_{xz} = \sigma_{zx} &= -\frac{Gb}{2\pi}\frac{y}{(-a-2d)^2+y^2)} \\ &= -\frac{Gb}{2\pi}\frac{y}{((2d+a)^2+y^2)} \end{aligned} \tag{14}$$

Similarly the total stress on surface 2 due to dislocations $\S_0$, $\S_1$, $\S_2$, $\S_3$ and $\S_4$ is

$$\begin{aligned} \sigma_{xz} = \sigma_{zx} &= -\frac{Gb}{2\pi}\frac{y}{((d-a)-(-2d))^2+y^2)} \\ &= -\frac{Gb}{2\pi}\frac{y}{((3d-a)^2+y^2)} \end{aligned} \tag{15}$$

Again we can see the stress $\sigma_{zx} \neq 0$ on the surfaces. So we again add two negative dislocations $\S_5$ at a distance $2d + a$ from surface 1 (outward) and $\S_6$ at a distance $3d - a$ from surface 2 (outward) so the net stress would be zero on the surfaces. Now the total stress on surface 1 due to dislocations $\S_0$, $\mathfrak{z}_1$, $\mathfrak{z}_2$, $\S_3$, $\S_4$ and $\mathfrak{z}_5$ is

$$\begin{aligned}\sigma_{xz} = \sigma_{zx} &= -\frac{Gb}{2\pi}\frac{y}{((2d+a)^2+y^2)} - \frac{G(-b)}{2\pi}\frac{y}{((-a-(-2d-2a))^2+y^2)} \\ &= -\frac{Gb}{2\pi}\frac{y}{((2d+a)^2+y^2)} + \frac{Gb}{2\pi}\frac{y}{((2d+a)^2+y^2)} \\ &= 0\end{aligned} \tag{16}$$

Similarly the total stress on surface 2 due to dislocations $\S_0$, $\mathfrak{z}_1$, $\mathfrak{z}_2$, $\S_3$, $\S_4$ and $\mathfrak{z}_6$ is

$$\begin{aligned}\sigma_{xz} = \sigma_{zx} &= -\frac{Gb}{2\pi}\frac{y}{((3d-a)^2+y^2)} - \frac{G(-b)}{2\pi}\frac{y}{((d-a)-(4d-2a))^2+y^2)} \\ &= -\frac{Gb}{2\pi}\frac{y}{((3d-a)^2+y^2)} + \frac{Gb}{2\pi}\frac{y}{((3d-a)^2+y^2)} \\ &= 0\end{aligned} \tag{17}$$

Again dislocation $\mathfrak{z}_5$ causes stress on surface 2 and $\mathfrak{z}_6$ on surface 1. Now the total stress σ_{zx} on surface 1 due to $\S_0$, $\mathfrak{z}_1$, $\mathfrak{z}_2$, $\S_3$, $\S_4$, $\mathfrak{z}_5$ and $\mathfrak{z}_6$ is

$$\begin{aligned}\sigma_{xz} = \sigma_{zx} &= -\frac{G(-b)}{2\pi}\frac{y}{(-a-2(2d-a))^2+y^2)} \\ &= \frac{Gb}{2\pi}\frac{y}{((4d+a)^2+y^2)}\end{aligned} \tag{18}$$

Similarly the total stress on surface 2 due to dislocations $\S_0$, $\mathfrak{z}_1$, $\mathfrak{z}_2$, $\S_3$, $\S_4$, $\mathfrak{z}_5$ and $\mathfrak{z}_6$ is

$$\begin{aligned}\sigma_{xz} = \sigma_{zx} &= -\frac{G(-b)}{2\pi}\frac{y}{((d-a)-2(-d-a))^2+y^2)} \\ &= \frac{Gb}{2\pi}\frac{y}{((3d+a)^2+y^2)}\end{aligned} \tag{19}$$

We can see a trend in the position of the image dislocations and that is

$$D\big|_{\mathfrak{z}_i} = -2(di+a)+2d,\ 2(di-a) \text{ and } D\big|_{\S_i} = \mp 2di$$

where, $i = 1, 2, \ldots \infty$. This is how we can rewrite eq. (5) considering all the image dislocations as,

$$\begin{aligned}\sigma_{xz} = \sigma_{zx} = -\frac{Gby}{2\pi}\Bigg[&\frac{1}{x^2+y^2} - \frac{1}{(x+2a)^2+y^2} - \frac{1}{\{x-2(d-a)\}^2+y^2} + \frac{1}{(x+2d)^2+y^2} + \frac{1}{(x-2d)^2+y^2} \\ &- \frac{1}{\{x+2(d+a)\}^2+y^2} - \frac{1}{\{x-2(2d-a)\}^2+y^2} + \frac{1}{(x+4d)^2+y^2} + \frac{1}{(x-4d)^2+y^2} \\ &- \frac{1}{\{x+2(2d+a)\}^2+y^2} - \frac{1}{\{x-2(3d-a)\}^2+y^2} + \frac{1}{(x+6d)^2+y^2} + \frac{1}{(x-6d)^2+y^2} - \cdots\Bigg]\end{aligned} \tag{20}$$

$$\sigma_{yz} = \sigma_{zy} = \frac{Gb}{2\pi}\left[\frac{x}{x^2+y^2} - \frac{x+2a}{(x+2a)^2+y^2} - \frac{x-2(d-a)}{\{x-2(d-a)\}^2+y^2} + \frac{x+2d}{(x+2d)^2+y^2} + \frac{x-2d}{(x-2d)^2+y^2}\right.$$
$$- \frac{x+2(d+a)}{\{x+2(d+a)\}^2+y^2} - \frac{x-2(2d-a)}{\{x-2(2d-a)\}^2+y^2} + \frac{x+4d}{(x+4d)^2+y^2} + \frac{x-4d}{(x-4d)^2+y^2}$$
$$\left. - \frac{x+2(2d+a)}{\{x+2(2d+a)\}^2+y^2} - \frac{x-2(3d-a)}{\{x-2(3d-a)\}^2+y^2} + \frac{x+6d}{(x+6d)^2+y^2} + \frac{x-6d}{(x-6d)^2+y^2} - \cdots\right] \tag{21}$$

We can write the above equations as the summation of infinite series as

$$\sigma_{xz} = \sigma_{zx} = -\frac{Gby}{2\pi}\left[\frac{1}{x^2+y^2} - \sum_{i=1}^{N}\left\{\frac{1}{\{x+2(di-d+a)\}^2+y^2} + \frac{1}{\{x-2(di-a)\}^2+y^2} - \frac{1}{(x+2di)^2+y^2} - \frac{1}{(x-2di)^2+y^2}\right\}\right] \tag{22}$$

$$\sigma_{yz} = \sigma_{zy} = \frac{Gb}{2\pi}\left[\frac{x}{x^2+y^2} - \sum_{i=1}^{N}\left\{\frac{x+2(di-d+a)}{\{x+2(di-d+a)\}^2+y^2} + \frac{x-2(di-a)}{\{x-2(di-a)\}^2+y^2} - \frac{x+di}{(x+2di)^2+y^2} - \frac{x-di}{(x-2di)^2+y^2}\right\}\right] \tag{23}$$

where N should be ∞.

Solution Verification

Equations (22) and (23) have infinite series summation but in practice we summed up to $N = 10^5$ to get stress $\sigma_{xz} = \sigma_{zx} < G \times 10^{-12}$ on the either surface. Figure 4 shows the maximum σ_{zx}/G value on either surface, i.e. the global surface maxima, versus N.

In figs. 5 and 6, we show the stress field over the finite plate with and without incorporating the image dislocations placing a positive dislocation at 0.1d from surface 1. As we see, in fig. 5(left) where the image dislocations are not incorporated (i.e. stress calculated using eq. (4b)) the σ_{zx} stress is not zero on either surface. But in fig. 5(right) σ_{zx} is essentially zero ($< G \times 10^{-12}$) on the either surface where image dislocations are incorporated in the stress calculation (i.e. stress calculated using eq. (22)). We also show the σ_{zy} stress plotted over the finite plate in fig. 6 (image dislocations incorporated or not). In fig. 7(a-b) we show the stress σ_{zx} along $y = 4b$ and $y = 10b$. As can be seen for a positive dislocation, the difference between the stresses calculated using eqs. (4b) and (22) is more on the surface near to the dislocation and it diminishes as the dislocation moves far from the surface which is quite intuitive. Equations (4b) and (22) should produce same stress for a point far from the dislocation. We show similar picture for stress σ_{zy} in fig. 7(c-d).

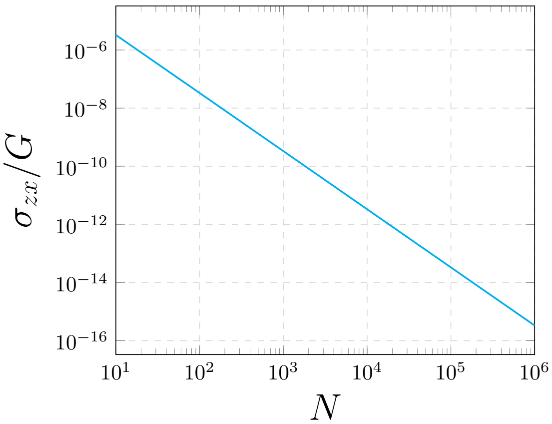


Figure 4: Solution convergence plot with N.

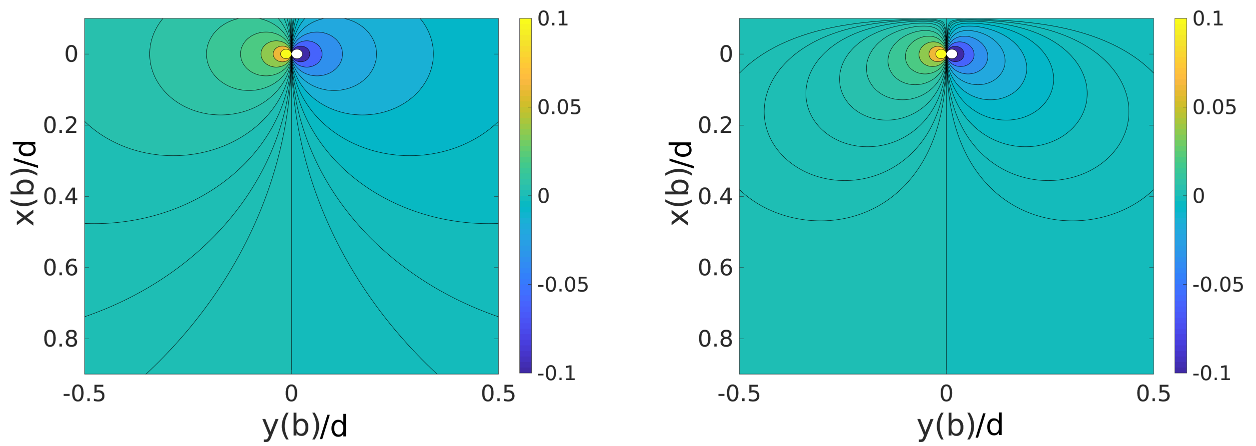


Figure 5: Contour plots of σ_{zx}/G. Positive dislocation $\S_0$ at $0.1d$ away from top surface. The figure on the right is showing the effect of image dislocations on the stress field

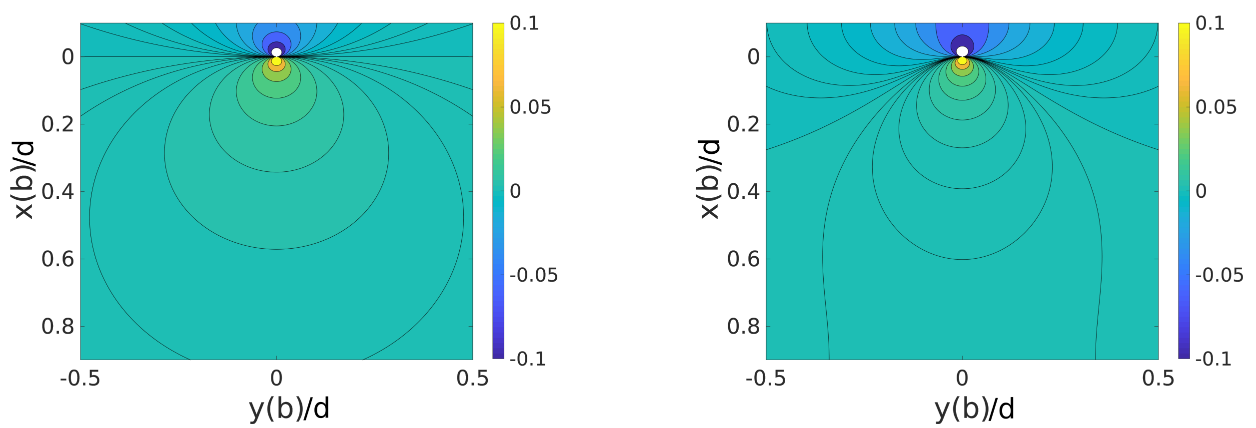
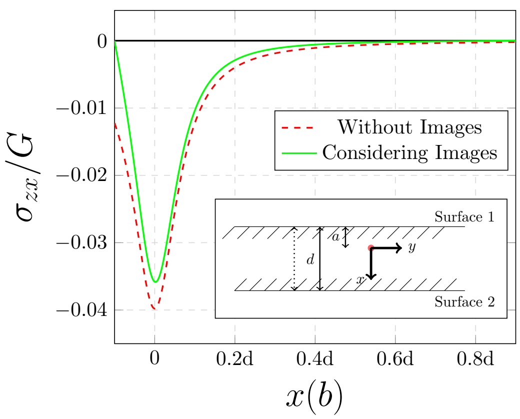
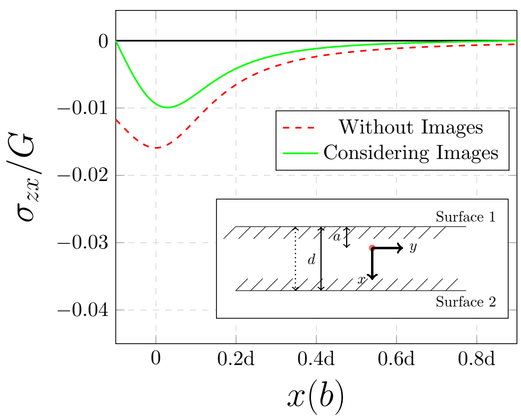


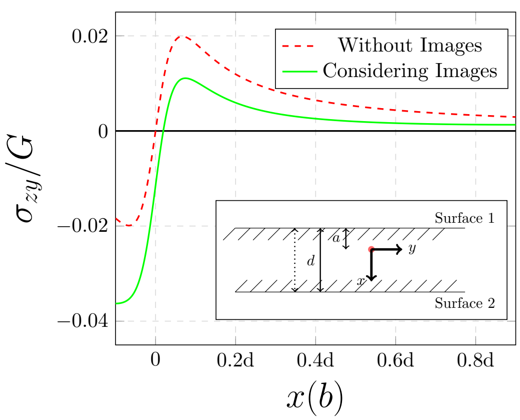
Figure 6: Contour plots of σ_{zy}/G. Positive dislocation $\S_0$ at $0.1d$ away from top surface. The figure on the right is showing the effect of image dislocations on the stress field



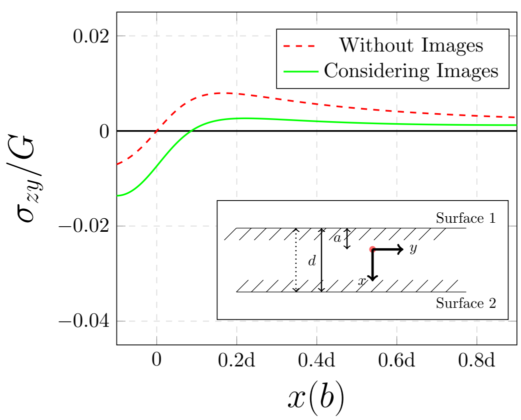
(a) along $y = 4b$



(b) along $y = 10b$



(c) along $y = 4b$



(d) along $y = 10b$

Figure 7: Positive dislocation $\S_0$ is at $0.1d$ away from top surface

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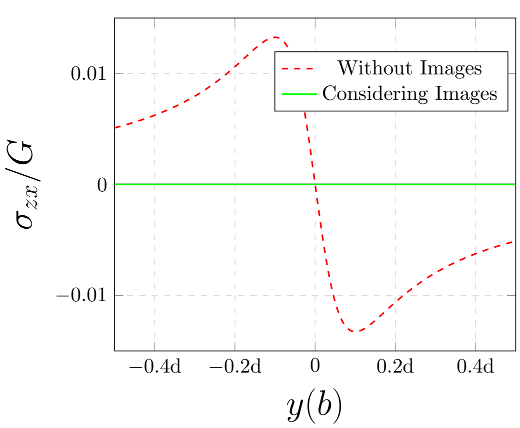
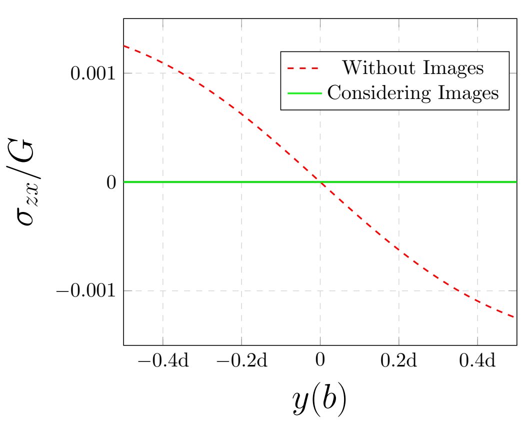


(a) Top surface



(b) Bottom surface

Figure 8: Stress σ_{zx} on the surfaces when positive dislocation $\S_0$ is at $0.1d$ away from top surface

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Finally, in fig. 8 we show the stress σ_{zy} on the free surfaces calculated using eqs. (4b) and (22) what confirms that we achieve the traction free surfaces when consider the image dislocations on either side of the plate.

Conclusion

The analytic formulation of stress field for infinite and semi-infinite medium do not satisfy the boundary conditions for the case of a finite medium. In this article, we have introduced the formulation of the stress field over a thin plate in the presence of image dislocations. Moreover, we have shown how the image dislocations are distributed. With the distributed image dislocations, we have confirmed the condition of traction-free surfaces using line and contour plots.

Acknowledgment

Department of Mechanical Engineering, University of New Mexico.
Website: http://www.unm.edu

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