Gödel logics with an operator shifting truth values

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Abstract

We consider GÃűdel logics extended by an operator whose semantics is given by $\mathfrak{I}(o(A)) = \min\{1, r + \mathfrak{I}(A)\}$.

The language of propositional GÃűdel logics $\mathcal{L}^{\mathrm{p}}$ consists of a countably infinite set Var of propositional variables and the connectives $\bot$, $\supset$, $\wedge$, $\vee$ with their usual arities. We will consider extensions by a unary connective o, by a unary connective $\triangle$ or by both. For any $r \in [0,1]$, a GÃűdel *r-interpretation* $\mathfrak{I}$ maps formulas to V such that $\mathfrak{I}(\bot) = 0$,

$$\mathfrak{I}(A \wedge B) = \min\{\mathfrak{I}(A), \mathfrak{I}(B)\},$$
$$\mathfrak{I}(A \vee B) = \max\{\mathfrak{I}(A), \mathfrak{I}(B)\},$$
$$\mathfrak{I}(A \supset B) = \begin{cases} 1 & \mathfrak{I}(A) \leq \mathfrak{I}(B), \\ \mathfrak{I}(B) & \mathfrak{I}(A) > \mathfrak{I}(B). \end{cases}$$

If the language contains o resp. $\triangle$, we additionally require

$$\mathfrak{I}(o(A)) = \min\{1, r + \mathfrak{I}(A)\},$$
$$\mathfrak{I}(\triangle(A)) = \begin{cases} 1 & \mathfrak{I}(A) = 1 \\ 0 & \mathfrak{I}(A) < 1. \end{cases}$$

Let G be some Hilbert-Frege style proof calculus that is sound and complete for propositional GÃűdel logics (without o and $\triangle$), e.g. take a proof system for intuitionistic logic, plus the schema of linearity $(A \supset B) \vee (B \supset A)$, see [3] or, alternatively, use one of the systems described in [4]. We prove that G enhanced by the axiom schemata $(\bot \prec o\bot) \supset (A \prec oA)$, $(\bot \leftrightarrow o\bot) \supset (A \leftrightarrow oA)$, and $o(A \supset B) \leftrightarrow (oA \supset oB)$ is sound and complete w.r.t. the above semantics. Generalizing ideas from [2], we also give an algorithm that constructs a proof for any valid formula. However, this semantics fails to have a compact entailment.

The above proof system can also be further combined with a proof system for $\triangle$, see [1], to yield a sound and complete calculus for the valid formulas in that language.

While the propositional fragment has quite a simple structure, we will show that first order GÃűdel logic enhanced by this ring operator is not recursively enumerable, using a technique by Scarpellini [5] employed for Łukasiewicz logic. This ring operator makes the borderline of similarities and contrasts between Łukasiewicz logic visible.

The situation changes if one interprets o, more generally, as a function with certain monotonicity properties.

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References

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