Kalpa Publications in Computing
Volume 2, 2017, Pages 85–94
ICRISET2017. International Conference on Research and Innovations in Science, Engineering &Technology. Selected Papers in Computing

Optimal homotopy analysis solution of fingero-imbibition phenomenon in homogeneous porous medium with magnetic fluid effect

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Abstract

The present paper discusses the fingero-imbibition phenomenon in a double phase displacement process through homogeneous porous medium with the involvement of a layer of magnetic fluid in the injected phase. This phenomenon has much importance in petroleum technology. The nonlinear partial differential equation governing this phenomenon with appropriate boundary conditions is solved by an optimal homotopy analysis method. The convergence of the solution is decided by minimizing discrete squared residual.

1 Introduction

When a porous medium filled with one phase (oil) is brought into the contact of another phase (water) which is preferentially wetting, then the spontaneous flow of the wetting phase into the medium and counter flow of the resident phase from the medium without any external force. This is known as imbibition phenomenon. Besides this if a phase (oil) contained in a porous medium is displaced by another phase of lesser viscosity, then instead of regular displacement of the whole front, protuberances may occur which shoot through the porous medium at relatively very great speed giving rise to the fingering phenomenon. This simultaneous occurrence of both phenomena was termed as fingero-imbibition by Verma.

Many researchers have discussed this problem from different viewpoints and solved by different methods: Mishra and Verma [14], Patel, Rabari and Bhathawala [19], Shah [26], Patel and Rabari [18],Shah and Verma [27], Parikh, Mehta and Pradhan [17], Verma and Rajput [31], Verma [30], Patel and Desai [20], Desai[4], Mehta [13] etc. In this paper, we have discussed the fingero-imbibition phenomenon through homogeneous porous medium with the involvement of a layer of magnetic fluid in the injected phase. In this work, the underlying assumptions are that the two phases are immiscible and the injected phase is less viscous as well as preferentially wetting. Also it is assumed that the magnetic field is directly proportional to the magnetic field intesity H and the macroscopic behaviour of fingers is governed by a statistical treatment. An approximate analytical solution of the governing nonlinear partial differential equation with suitable boundary conditions has been obtained by optimal homotopy analysis method.

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2 Formulation of the problem

It is considered that a finite porous matrix with native phase (n) is completely surrounded by an impermeable surface except for one end of the matrix which is labelled as the imbibition face $x = 0$ and this end is exposed to an adjacent formation of injected phase (i) which involves a thin layer of magnetic fluid. This arrangement gives rise to a displacement process in which the injection of the fluid (i) is initiated by imbibition and the consequent displacement of native phase (n) produces proturberances (fingers).

For the mathematical formulation, we assume that the Darcy's law is valid for the investigated flow system and the macroscopic behaviour of fingers is governed by a stastical treatment.[29] In this case, the saturation ($S_i(x,t)$) of the injected phase (i) is defined as the average cross-sectional area occupied by it at the level (x) and time (t) and thus the saturation of the displacing phase in the porous medium represents the average cross-sectional area occupied by the fingers. Let injected phase (i) and native phase (n) be two immiscible phases governed by Darcy's law and the velocities of water and oil be expressed as [3, 15]

$$V_i = -\frac{k_i}{\mu_i} K \left[\frac{\partial p_i}{\partial x} + \alpha H \frac{\partial H}{\partial x} \right] \tag{1}$$

$$V_n = -\frac{k_n}{\mu_n} K \frac{\partial p_n}{\partial x} \tag{2}$$

where

$$\alpha = \mu_0 \chi + \frac{16 \pi \mu_0 \chi^2 r^3}{9 \left(l+2\right)^3} [27] \tag{3}$$

V_i and V_n are the velocities of water and oil respectively, K is the permeability of the homogeneous porous medium which is constant, k_i and k_n are the relative permeabilities of water and oil respectively, μ_i and μ_n are the constant viscosities of water and oil respectively, p_i and p_n are the pressures of water and oil respectively, H is the magnetic field intensity, μ_0 is permeability (magnetic) of free space, l is the centre to centre distance of magnetic fluid particles, r is the radius of magnetic fluid particles and χ is susceptibility.

The additional term $\alpha H \frac{\partial H}{\partial x}$ on the right hand side of (1) represents the additional pressure gradient exerted due to the presence of a layer of magnetic fluid in the displacing phase (i).

Here we consider densities of both phases as constant. Hence the mass conservation laws for both phases of the mixture have the form [3]:

$$P \frac{\partial S_i}{\partial t} + \frac{\partial V_i}{\partial x} = 0 \tag{4}$$

$$P \frac{\partial S_n}{\partial t} + \frac{\partial V_n}{\partial x} = 0 \tag{5}$$

where P is the porosity of the medium regarded as constant.

In finger-imbibition, the velocities of injected and native phases are in opposite direction. According to Scheidegger [24]

$$V_i = -V_n \tag{6}$$

The capillary pressure p_c is a function of the phase saturation. According to Scheidegger [24], it may be written as

$$p_c(S_i) = p_n - p_i \tag{7}$$

and

$$p_c = -\beta S_i \tag{8}$$

where β is a constant.

For definiteness of the mathematical analysis, the relationship between phase saturation and relative permeability as given by Scheidegger and Johnson [25] is used here.

$$k_i = S_i \quad and \quad k_n = S_n \tag{9}$$

From the definition of phase saturation,

$$S_i + S_n = 1 \tag{10}$$

Using (1) and (2) in (6), we have

$$\frac{k_i}{\mu_i} K \left[\frac{\partial p_i}{\partial x} + \alpha H \frac{\partial H}{\partial x} \right] + \frac{k_n}{\mu_n} K \frac{\partial p_n}{\partial x} = 0 \tag{11}$$

Using (7) and simplyfying, we have

$$\frac{\partial p_i}{\partial x} = -\frac{k_i \mu_n \alpha H \frac{\partial H}{\partial x} + k_n \mu_i \frac{\partial p_c}{\partial x}}{k_i \mu_n + k_n \mu_i} \tag{12}$$

On substituting the value of (12) in (1) and simplyfying, we get

$$V_i = K \frac{k_i k_n}{k_i \mu_n + k_n \mu_i} \left[\frac{\partial p_c}{\partial x} - \alpha H \frac{\partial H}{\partial x} \right] \tag{13}$$

Also it is assumed that

$$\frac{k_i k_n}{k_i \mu_n + k_n \mu_i} \approx \frac{k_n}{\mu_n} [24]$$

Hence (13) reduces to

$$V_i = K \frac{k_n}{\mu_n} \left[\frac{\partial p_c}{\partial x} - \alpha H \frac{\partial H}{\partial x} \right] \tag{14}$$

Using (14) in (4), we get

$$P \frac{\partial S_i}{\partial t} + \frac{\partial}{\partial x} \left[K \frac{k_n}{\mu_n} \left[\frac{\partial p_c}{\partial x} - \alpha H \frac{\partial H}{\partial x} \right] \right] = 0 \tag{15}$$

Using (8), (9) and (10) into (15), we obtain

$$P \frac{\partial S_i}{\partial t} - \frac{\beta K}{\mu_n} \frac{\partial}{\partial x} \left[(1 - S_i) \frac{\partial S_i}{\partial x} \right] + \frac{K \alpha H}{\mu_n} \frac{\partial H}{\partial x} \frac{\partial S_i}{\partial x} - \frac{K \alpha (1 - S_i)}{\mu_n} \left[H \frac{\partial^2 H}{\partial x^2} + \left(\frac{\partial H}{\partial x} \right)^2 \right] = 0 \tag{16}$$

Here we consider the auxiliary magnetic field H in the x-direction only,

$$H = \frac{\lambda}{x^n}[27] \tag{17}$$

where λ is a constant parameter and n is an integer. Using (17) for $n = -1$ in (16), we obtain

$$P\frac{\partial S_i}{\partial t} - \frac{\beta K}{\mu_n}\frac{\partial}{\partial x}\left[(1-S_i)\frac{\partial S_i}{\partial x}\right] + \frac{Kx\alpha\lambda^2}{\mu_n}\frac{\partial S_i}{\partial x} - \frac{K\alpha\lambda^2(1-S_i)}{\mu_n} = 0 \tag{18}$$

Using dimensionless variables

$$X = \frac{x}{L},\ \ T = \frac{\beta K t}{P\mu_n L^2},$$

(18) reduces to

$$\frac{\partial S_i}{\partial T} + \frac{\partial}{\partial X}\left[S_i\frac{\partial S_i}{\partial X}\right] + C\frac{\partial}{\partial X}\left[S_i X\right] - \frac{\partial^2 S_i}{\partial x^2} - C = 0 \tag{19}$$

where

$$C = \frac{\alpha\lambda^2 L^2}{\beta}$$

Eq.(19) is desired nonlinear partial differential equation of motion for the flow of two immiscible phases in homogeneous medium with effect of magnetic fluid.

Let at the common interface, the saturation of injected water be linear function of time, that is

$$S_i(0,T) = aT \quad for\ T > 0 \tag{20}$$

where a is a constant.

Since, it is assumed that the porous medium is completely surrounded by an impermeable surface except for one end, we consider

$$\frac{\partial S_i}{\partial X}(1,T) = 0 \quad for\ T > 0 \tag{21}$$

We solve equation (19) together with boundary conditions (20) and (21) using optimal homotopy analysis method.

3 Solution of the problem using Optimal Homotopy Analysis Method

We choose

$$S_{i_0}(X,T) = aT + Xe^{-X} \tag{22}$$

as the initial approximation of $S_i(X,T)$ which satisfies boundary conditions (20) and (21).

Besides we choose the auxiliary linear operator as

$$\mathcal{L}[\phi(X,T;q)] = \frac{\partial^2\phi(X,T;q)}{\partial X^2} + \frac{\partial\phi(X,T;q)}{\partial X} \tag{23}$$

with the property

$$\mathcal{L}[f] = 0 \quad when\ f = 0. \tag{24}$$

Furthermore, based on governing equation (19), we define a nonlinear operator as

$$\mathcal{N}[\phi(X,T;q)] = \frac{\partial\phi(X,T;q)}{\partial T} + \phi(X,T;q)\frac{\partial^2\phi(X,T;q)}{\partial X^2} - \frac{\partial^2\phi(X,T;q)}{\partial X^2} + \left\{\frac{\partial\phi(X,T;q)}{\partial X}\right\}^2 + C\{\phi(X,T;q)-1\} + CX\frac{\partial\phi(X,T;q)}{\partial X} \tag{25}$$

Let c_0 denote a nonzero auxiliary parameter. According to Liao [8], the zeroth order deformation equation is

$$(1-q)\mathcal{L}[\phi(X,T;q) - S_{i_0}(X,T)] = c_0 q H(X,T)\mathcal{N}[\phi(X,T;q)] \tag{26}$$

where $q \in [0,1]$ is the embedding parameter, $H(X,T)$ is nonzero auxiliary function and $\phi(X,T;q)$ is an unknown function. Obviously, when $q = 0$ and $q = 1$, we have from (24) and (26),

$$\phi(X,T;0) = S_{i_0}(X,T) \tag{27}$$

and

$$\phi(X,T;1) = S_i(X,T) \tag{28}$$

Therefore, $\phi(X,T;q)$ varies from the initial approximation $S_{i_0}(X,T)$ to the solution $S_i(X,T)$ of the equation (19) as the embedding parameter q increases from 0 to 1. Obviously, $\phi(X,T;q)$ is determined by the auxiliary linear operator $\mathcal{L}$, the initial guess $S_{i_0}(X,T)$ and the auxiliary parameter c_0. We have great freedom to select all of them. Assuming that all of them are so properly chosen that the Taylor series

$$\phi(X,T;q) = S_{i_0}(X,T) + \sum_{m=1}^{\infty} S_{i_m}(X,T)q^m \tag{29}$$

exists and converges at $q = 1$, we have the homotopy-series solution

$$S_i(X,T) = S_{i_0}(X,T) + \sum_{m=1}^{\infty} S_{i_m}(X,T) \tag{30}$$

where

$$S_{i_m}(X,T) = \frac{1}{m!}\frac{\partial^m\phi(X,T;q)}{\partial q^m}\bigg|_{q=0} \tag{31}$$

Differentiating the zeroth order deformation equation (26) m times with respect to the embedding parameter q and then dividing by $m!$ and finally setting $q = 0$, we have the so called high order deformation equation

$$\mathcal{L}[S_{i_m}(X,T) - \chi_m S_{i_{m-1}}(X,T)] = c_0 H(X,T)\mathcal{R}_m(X,T) \tag{32}$$

subject to the boundary conditions

$$S_{i_m}(0,T) = 0, \quad \frac{\partial S_{i_m}}{\partial X}(1,T) = 0, \quad m \geq 1 \tag{33}$$

where

$$\mathcal{R}_m(X,T) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}[\phi(X,T;q)]}{\partial q^{m-1}}\bigg|_{q=0} \tag{34}$$

and

$$\chi_m = \begin{cases} 0 & \text{if } m \leq 1, \\ 1 & \text{if } m > 1. \end{cases} \tag{35}$$

It is very important to emphasize that (32) is linear for all $m \geq 1$. Thus we convert the original nonlinear problem into an infinite sequence of linear subproblems governed by (32). For simplicity, assume $H(X,T) = 1$. The solution of the mth order deformation equation (32) for $m \geq 1$ is

$$S_{i_m}(X,T) = \chi_m S_{i_{m-1}}(X,T) + c_0\mathcal{L}^{-1}[\mathcal{R}_m(X,T)] + C_1 X + C_2 \tag{36}$$

where C_1 and C_2 are constants or functions of T and we determine them using boundary conditions (33).The equations (36) can be easily solved by symbolic computation software such as Mathematica. Hence the approximate analytical solution to the given nonlinear problem takes the following form:

$$\begin{aligned} S_i(X,T) &= aT + Xe^{-X} - c_0 \\ &\quad - ec_0\left(1 - e^{-X}\right)\left[a + aCT + \frac{C}{e} + \frac{1}{e}\left(\frac{1}{2} - \frac{C}{3}\right) - \frac{3}{e^2} - \frac{aT}{2e}\right] \\ &\quad + c_0\Bigg[aX + aCTX - \left(X - \frac{X^2}{2} - \frac{CX^3}{3}\right)e^{-X} + \left(X^2 + X + 1\right)e^{-2X} \\ &\quad - \frac{aT\left(X^2 - 2X\right)e^{-X}}{2}\Bigg] + \cdots \end{aligned} \tag{37}$$

which represents the saturation of injected water $S_i(X,T)$ in homogeneous porous medium at distance X and time T.

This solution expression contains the auxiliary parameter c_0 which is called the convergence-control parameter and is employed to control the convergence of the solution.

According to Liao[11], the discrete squared residual at the mth order of approximation is

$$E_m = \frac{1}{(M+1)(N+1)}\sum_{i=0}^{M}\sum_{j=0}^{N}\left\{\mathcal{N}\left[\sum_{n=0}^{m} S_{i_n}\left(\frac{i}{M},\frac{j}{N}\right)\right]\right\}^2 \tag{38}$$

The value of c_0 can be optimally identified from the condition

$$\frac{dE_m(c_0)}{dc_0} = 0 \tag{39}$$

This optimization approach for obtaining the parameter c_0 has been applied recently to a number of problems for nonlinear ordinary and partial differential equations [9, 10, 16, 2, 23, 21, 22, 5, 6, 1, 12, 32, 28, 7]. Fig. 1 shows the curve of discrete squared residual at the 5th order of approximation E_5 versus c_0.

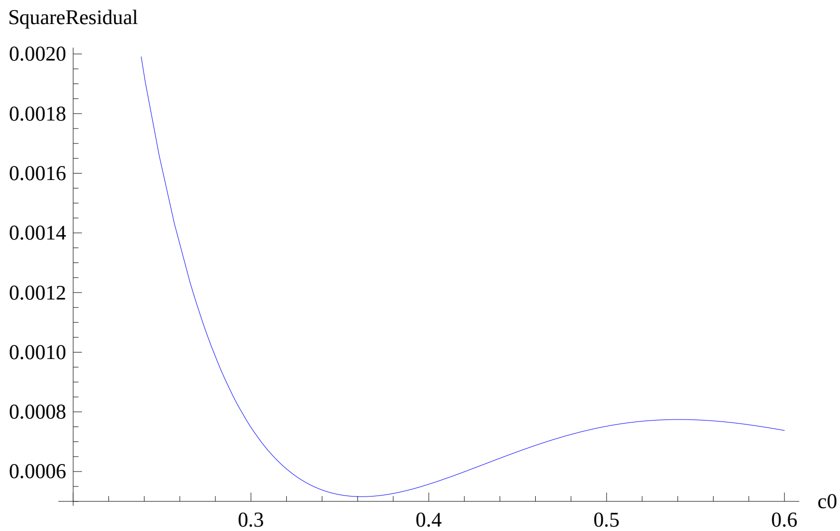

Fig. 1: E_5 versus c_0.

Using Mathematica, we find that square residual E_5 has its minimum value 5.15788×10^{-4} at $c_0 = 0.362293$ for $M = N = 50$ which can be seen in Fig. 1 also.

4 Numerical and Graphical Representation

We obtain the numerical and graphical representations of the solution $S_i(X,T)$ up to 5th order approximation using $c_0 = 0.362293$. Also we use the following values of parameters:

$$L = 1.0m,\ \beta = 6895N/m^2,\ \mu_0 = 4\pi \times 10^{-7}H/m,$$

$$\chi = 0.2805031,\ \lambda = 1.0,\ r = 1.03281 \times 10^{-9}m,\ l = 20 \times 10^{-9}m,$$

$$C = 5.10515 \times 10^{-11},\ a = 0.001,\ \alpha = 3.52 \times 10^{-7}H/m.$$

Numerical values of the saturation of injected water obtained by the optimization approach up to 5th order approximation are given in Table 1. Fig. 2 represents the graph of saturation of injected water, $S_i(X,T)$, versus distance X for fixed values of time $T = 200, 250, 300, \cdots, 600$. Fig. 3 represents the graph of saturation of injected water, $S_i(X,T)$, versus time T for fixed values of distance $X = 0.1, 0.2, \cdots, 1$. We use numerical values of Table 1 for Fig. 2 and Fig. 3.

Table 1: Numerical values of the saturation of injected water.

T	$X = 0.1$	$X = 0.2$	$X = 0.3$	$X = 0.4$	$X = 0.5$	$X = 0.6$	$X = 0.7$	$X = 0.8$	$X = 0.9$	$X = 1$
200	0.201311	0.202704	0.204440	0.206539	0.208856	0.211174	0.213261	0.214916	0.215982	0.216353
250	0.253108	0.256342	0.259864	0.263593	0.267314	0.270764	0.273700	0.275926	0.277309	0.277780
300	0.305367	0.310882	0.316581	0.322272	0.327661	0.332443	0.336364	0.339248	0.340995	0.341575
350	0.358148	0.366436	0.374748	0.382767	0.390113	0.396443	0.401502	0.405138	0.407297	0.408002
400	0.411515	0.423124	0.434531	0.445279	0.454900	0.463013	0.469372	0.473863	0.476488	0.477333
450	0.465536	0.481072	0.496106	0.510022	0.522263	0.532416	0.540252	0.545708	0.548855	0.549857
500	0.520283	0.540416	0.559658	0.577223	0.595459	0.604929	0.614432	0.620971	0.624701	0.625877
550	0.575833	0.601298	0.625383	0.647119	0.665756	0.680842	0.692218	0.699967	0.704345	0.705714
600	0.632268	0.663868	0.693490	0.719962	0.742439	0.760462	0.773930	0.783025	0.788121	0.789703

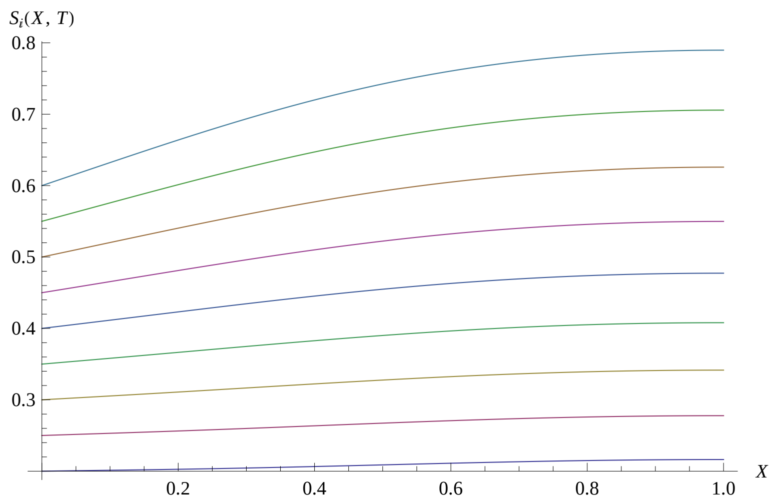

Fig. 2: $S_i(X,T)$ versus X for fixed values of times
$T = 200(lowermost\ graph), 250, 300, 350, \cdots, 600(uppermost\ graph)$

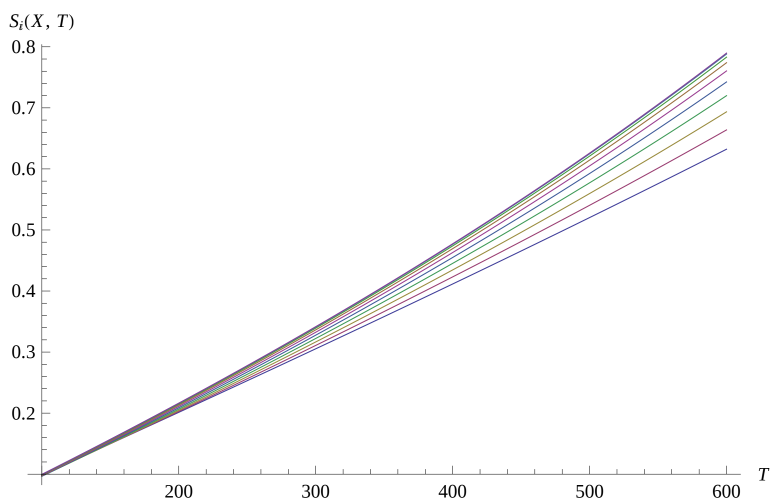

Fig. 3: $S_i(X,T)$ versus T for fixed values of distance
$X = 0.1(lowermost\ graph), 0.2, \cdots, 1(uppermost\ graph)$

5 Conclusion

In this paper we have used the optimal homotopy analysis method to obtain the approximate analytical solution for the fingero-imbibition phenomenon in homogeneous porous medium with magnetic fluid effect. The optimal homotopy analysis method provides us a simple way to adjust and control the convergence of the series solution by choosing proper value of auxiliary parameter. From Fig. 2, we can see that the saturation of injected water increases when the distance increases for fixed value of time. Also the saturation of water increases when time increases for fixed value of distance which can be observed in Fig. 3. This is consistent with real phenomenon.

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