Effect algebras with state operator

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Effect algebras have been introduced by Foulis and Bennett [2] (see also [3, 4] for equivalent definitions) for modeling unsharp measurements in quantum mechanical systems [5]. They are a generalization of many structures which arise in the axiomatization of quantum mechanics (Hilbert space effects [7]), noncommutative measure theory and probability (orthomodular lattices and posets, [6]), fuzzy measure theory and many-valued logic (MV-algebras [10, 9]).

A state, as an analogue of a probability measure, is a basic notion in algebraic structures used in the quantum theories (see e.g., [8]), and properties of states have been deeply studied by many authors.

In MV-algebras, states as averaging the truth value were first studied in [11]. In the last few years, the notion of a state has been studied by many experts in MV-algebras, e.g, [13, 12].

Another approach to the state theory on MV-algebras has been presented recently in [15]. Namely, a new unary operator was added to the MV-algebras structure as an internal state (or so-called state operator). MV-algebras with the added state operator are called state MV-algebras. The idea is that an internal state has some properties reminiscent of states, but, while a state is a map from an MV-algebra into $[0, 1]$, an internal state is an operator of the algebra. State MV-algebras generalize, for example, Hájek's approach [14] to fuzzy logic with modality Pr (interpreted as *probably* with the following semantic interpretation: The probability of an event a is presented as the truth value of $Pr(a)$. For a more detailed motivation of state MV-algebras and their relation to logic, see [15].

In [1], the notion of a state operator was extended from MV-algebras to the more general frame of effect algebras. A state operator is there defined as an additive, unital and idempotent operator on E. A state operator on E is called strong, if it satisfies the additional condition

$$\tau(\tau(a) \wedge \tau(b)) = \tau(a) \wedge \tau(b) \text{ whenever } \tau(a) \wedge \tau(b) \text{ exists in } E. \tag{1}$$

Since MV-algebras form a special subclass of effect algebras, so-called MV-effect algebras, it was shown that the definition of a state operator on effect algebras coincides with the original definition on MV-algebras if and only if the state operator is strong. Moreover, if τ is faithful, i.e., $\tau(a) = 0$ implies $a = 0$, then property (1) is automatically satisfied.

In the present paper, we show that state operators on an effect algebra E are related with states on E in the following way: (1) every state on E induces a state operator on the tensor product $[0, 1] \otimes E$. (2) If E admits an ordering set of states, then every state operator on E induces a state on E. We study state operators mainly on convex effect algebras. Convex effect algebras as effect algebras with an additional convexity structure were introduced and studied in [17, 16]. It was proved in [17], that every convex effect algebra is isomorphic with the interval $[0, u]$ in an ordered real linear space (V, V^+), where u is an order unit. Moreover, (V, u) is an order unit space, i.e. u is an archimedean order unit, if and only if E admits an ordering set of states [16]. Using the tensor product of an effect algebra with the interval $[0, 1]$ of reals, we

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show that every effect algebra E admitting at least one state, can be embedded into a convex one. Moreover, a state operator on E extends to a state operator on its convex envelope. It is therefore not too restrictive to concentrate our interest on convex effect algebras with state operators. We show that a state operator on a convex effect algebra is an affine mapping which extends to a linear, positive and idempotent mapping of the corresponding ordered linear space into itself. Moreover, the state operator is faithful if and only if its extension is faithful.

A prototype of an effect algebra is the set of Hilbert space effects $\mathcal{E}(H)$, i.e., self-adjoint operators between the zero and identity operator on a (complex) Hilbert space H with respect to the usual ordering of self-adjoint operators. The partial operation $\oplus$ is defined as the usual operator sum of two effects whenever this sum is also an effect, and the effect algebra ordering coincides with the original one we started with. This effect algebra is naturally convex. It plays an important role in quantum measurement theory, because the most general quantum observables, positive operator valued measures (POVMs), have their ranges in it [5]. On the set $\mathcal{B}(H)$ of the bounded operators on H, we consider the von Neumann [19] and Lüders [18] conditional expectations, and we show that their restrictions to $\mathcal{E}(H)$ are faithful, hence strong state operators. Motivated by these examples, we study relations between state operators and conditional expectations on so-called JC-effect algebras.

Recall that a JC-algebra $\mathcal{J}$ is a norm-closed real vector subspace of bounded self-adjoint operators on a Hilbert space $\mathcal{H}$, closed under the Jordan product $a\circ b = \frac{1}{2}(ab+ba)$. A JC-algebra is called a JW-algebra if it is closed in the weak topology [22]. We study JC-algebras containing a unit element 1, and the interval $[0,1]$ is then called a *JC-effect algebra*. Let $\tau : \mathcal{E}(\mathcal{J}) \to \mathcal{E}(\mathcal{J})$ be a state operator. Since $\mathcal{E}(\mathcal{J})$ is the $[0,1]$ interval in the ordered vector space $(\mathcal{J}, \mathcal{J}^+)$, τ extends to a linear, positive idempotent and unital mapping $\tilde{\tau} : \mathcal{J} \to \mathcal{J}$. Such maps were studied in [21].

We call a state operator $\tau : \mathcal{E}(\mathcal{J}) \to \mathcal{E}(\mathcal{J})$ a *conditional expectation* iff it has the property $\tau(\tau(a)b\tau(a)) = \tau(a)\tau(b)\tau(a)$ for all $a, b \in \mathcal{E}(\mathcal{J})$. We show that a state operator τ on $\mathcal{E}(\mathcal{J})$ is a conditional expectation iff its range is a JC-sub-effect algebra of $\mathcal{E}(\mathcal{J})$, and any faithful state operator is a conditional expectation. We also show that if the JC-effect algebra is the unit interval in $C(X;\mathbb{R})$, the set of real-valued continuous functions on a compact Hausdorff space X, then a state τ operator is a conditional expectation iff τ is strong.

In the probability theory on MV-algebras, an MV-conditional expectation on a σ-MV-algebra M with respect to a σ-MV-subalgebra N of M in a σ-additive state m on M was introduced in [20]. We study relations between the MV-conditional expectations and state operators on convex σ-MV-algebras. Since a convex σ-MV-algebra M is isomorphic with the interval $[0,1]$ in $C(X;\mathbb{R})$ ([8, Theorem 7.3.12], a state operator τ on M is a conditional expectation iff τ is strong.

We show that an MV-conditional expectation induces a strong state operator on the quotient of M with respect to the kernel of the state m.

On the other hand, a strong σ-additive state operator (hence a conditional expectation) τ on M induces an MV-conditional expectation on M with respect to the sub-MV-algebra equal to the range of τ and in states of the form $s \circ \tau$, where s is any σ-additive state.

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